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Bäcklund transformations for finite-dimensional integrable systems: a geometric approach

Vadim Kuznetsov [a], Pol Vanhaecke [b,*]

[a] *Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK*

[b] *Département de Mathématiques, Université de Poitiers, UFR Sciences SP2MI, Téléport 2, Boulevard Marie et Pierre Curie, BP 30179, F-86962 Futuroscope Chasseneuil Cedex, France*

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Abstract

We present a geometric construction of Bäcklund transformations and discretizations for a large class of algebraic completely integrable systems. To be more precise, we construct families of Bäcklund transformations, which are naturally parameterized by the points on the spectral curve(s) of the system. The key idea is that a point on the curve determines, through the Abel–Jacobi map, a vector on its Jacobian which determines a translation on the corresponding level set of the integrals (the generic level set of an algebraic completely integrable systems has a group structure). Globalizing this construction we find (possibly multi-valued, as is very common for Bäcklund transformations) maps which preserve the integrals of the system, they map solutions to solutions and they are symplectic maps (or, more generally, Poisson maps). We show that these have the spectrality property, a property of Bäcklund transformations that was recently introduced. Moreover, we recover Bäcklund transformations and discretizations which have up to now been constructed by ad hoc methods, and we find Bäcklund transformations and discretizations for other integrable systems. We also introduce another approach, using pairs of normalizations of eigenvectors of Lax operators and we explain how our two methods are related through the method of separation of variables.

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* Corresponding author.
E-mail addresses: vadim@amsta.leeds.ac.uk (V. Kuznetsov), pol.vanhaecke@mathlabo.univ-poitiers.fr (P. Vanhaecke).

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1. Introduction

The theory of integrable maps got a boost, if was not virtually (re)started, a decade ago, when Veselov developed a theory of Lagrange correspondences [24–26]. Roughly speaking, *integrable maps* (also called *integrable Lagrange correspondences*) are symplectic multi-valued mappings which have enough integrals of motion, this definition being a proper analog of the classical Liouville integrability. In the main examples, studied by him and later by others, the integrable maps that are constructed are time-discretizations of some classical Liouville integrable systems (such as the Neumann system, the geodesic flow on an ellipsoid, the Euler–Manakov top, the Toda lattice, Calogero–Moser systems and other integrable families), see, for instance, [3,4,10–12,14,17,18]. It follows that these symplectic maps associate to a given solution of the integrable system a new solution, a property reminiscent of Bäcklund transformations for soliton equations; thus, one speaks in this context often of a *Bäcklund transformation* for the integrable system.

Recently [12] a new property of *spectrality* of Bäcklund transformations was introduced. Namely, it was observed that when one searches for the simplest Bäcklund transformations of an integrable system, then one actually finds a one-dimensional family $\{B_\lambda | \lambda \in \mathbb{C}\}$ of them and, most importantly, that the variable μ which is essentially the conjugate[1] to λ is bound to λ by the equation of an algebraic curve (dependent on the integrals), which is precisely the curve that appears in the linearization (integration) of the integrable system. The term *spectrality* stems from the fact that these curves arise most often as spectral curves, e.g. when the vector fields of the integrable system are given by Lax equations.

The purpose of this paper is to present a systematic construction of Bäcklund transformations for a large class of integrable systems which includes most classical integrable systems and many new ones. Some of the flavors of our methods and results are as follows:

1. Our Bäcklund transformations B_λ are given by explicit formulas rather than implicit equations.
2. We find big families of maps: one can let the parameter λ vary from one level manifold of the integrals to another.
3. They are symplectic (or Poisson) with respect to several compatible symplectic (or Poisson) structures.
4. Although our maps are n-valued (two-valued in the examples), they lead to single-valued maps on any level manifold of the integrals.
5. The resulting multi-point maps will discretize a family of flows of the integrable system (and not just a particular one).
6. The maps (and their iterates) are defined over an extension field $\mathbb{Q}(\sqrt{p})$ of $\mathbb{Q}$, where p depends on the initial conditions (values of the integrals) only.

These properties imply that our Bäcklund transformations are very well suited as symplectic integrators for the underlying integrable systems (see [15]).

Our methods will be restricted to those integrable systems (defined over $\mathbb{C}$) which have "good" algebraic geometric properties. These systems, baptized algebraic completely

[1] Since B_λ is symplectic it is given by a canonical transformation F_λ, which depends on λ. The conjugate of λ is given by $\partial F_\lambda / \partial \lambda$.

integrable systems (a.c.i. systems) by Adler and van Moerbeke (see [1]) have algebraic integrals and Poisson structures, and the generic common level set of the integrals is an affine part of a complex algebraic torus (*Abelian variety*) on which the flow of the integrable vector fields evolves linearly. A Bäcklund transformation B_λ, as defined above, will leave each such level set invariant. But it is well known that Abelian varieties are rigid in the sense that a holomorphic map between Abelian varieties is a group automorphism, followed by a translation. The automorphism group of an Abelian variety being finite, B_λ consists of a pure translation if it depends effectively on λ and is the identity map for some value of λ. If one wants to construct Bäcklund transformations, one may therefore be tempted to prescribe for every level set a g-dimensional vector (g is the dimension of the level set) but one is certainly doomed to fail when one wants to write down explicitly in algebraic coordinates the map which results from a translation over this family of vectors.

When the Abelian varieties that appear in the a.c.i. system are Jacobians then there is a special family of translations, given by pairs of points on the underlying algebraic curve (the Jacobian of an algebraic curve of genus g is a g-dimensional Abelian variety). Using the *explicit* correspondence between the points of phase space and the points on a Jacobian (represented either as divisors or line bundles on the underlying curve) we write down the meromorphic function on the curve that realizes the linear equivalence as follows:

$$\mathcal{D} + P \sim_l \tilde{\mathcal{D}} + Q, \tag{1}$$

where P and Q are the two points on the curve and the divisors $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are the two divisors which correspond to a generic point on phase space and its image under the Bäcklund transformation (this function is unique up to a constant factor). When expressed in terms of the phase variables this provides us with the map that gives the desired translation over the element $[P - Q]$ of the Jacobian. If one fixes one of the points, say Q, one recovers a one-dimensional family of maps, indexed by a point P on the curve. Notice that we can vary the points from one Jacobian to the other; however, there is an unavoidable monodromy problem, which makes that the points P and Q may get interchanged (leading to precisely the opposite vector, hence the inverse Bäcklund transformation), thus leading to a two-valued map.

For example, for the (g-dimensional) Mumford system (see [21]), phase space is the affine space of all matrices

$$L(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix},$$

where u, v and w are polynomials in x with u and w monic and

$$\deg v(x) < \deg u(x) = \deg w(x) - 1 = g.$$

The family of maps that we construct are given by the similarity transformation

$$L(x) \mapsto M(x)L(x)M^{-1}(x) \tag{2}$$

with

$$M(x) = \begin{pmatrix} \beta & x - \lambda_f + \beta^2 \\ 1 & \beta \end{pmatrix}, \tag{3}$$

where $\beta = (\mu_f - v(\lambda_f))/u(\lambda_f)$ and (λ_f, μ_f) is the chosen point P (dependent on f) on the spectral curve $y^2 = f(x) = -\det L(x)$ and Q is the point at infinity of this curve. It is easy to see that these maps satisfy properties 1, 2, 4 and 6 above.

By direct computation we find, in each example, a large class of Poisson maps. In the case of the Mumford system for example we show that when P varies such that its first coordinate depends on the Casimirs of the Poisson structure only, then we get a Poisson map, thereby establishing property 3.

When the level manifolds of the a.c.i. system are not Jacobians then they are, in all known examples where the integrals are known explicitly, covers of Jacobians, and we get Bäcklund transformations in an implicit form, i.e. we get Lagrangian correspondences as in Veselov's original paper [24]. See Section 3.5 for an example. The same applies to g.a.c.i. systems (a.c.i. in the generalized sense, see [2]). When the level manifolds are more general Abelian algebraic groups (a.c.i. in the sense of Mumford) then they are extensions of Abelian varieties by one or more copies of $\mathbb{C}^*$ and our technique again applies, see Sections 3.2 and 3.3 for examples.

When we let $Q \to P$ then we find at the first order a vector field which is constant on every level manifold because Q and P depend on the integrals only, so their restrictions to these level manifolds are linear combinations of the integrable vector fields. They need not be globally Hamiltonian, but we will present in our examples one-parameter families of points (P, Q) which lead to precisely the integrable vector fields of the a.c.i. system (property 5). In these cases the Bäcklund transformations should be considered as discretizations of the integrable system. Since these Bäcklund transformations commute, by construction, one may think of these as defining a discrete analog of an a.c.i. system.

Below we will also present another, but related, technique to construct the maps that represent translations on the level manifolds (assumed to be affine parts of Jacobians) of the integrals. For this it is assumed that phase space is given by Lax operators. We choose two different normalizations of the eigenvectors of the Lax operator, leading to two different separations of variables. This results in a map which is identical to the one that we constructed before. The reason is that the two different normalizations, which lead to linearly equivalent divisors, are chosen such that each has a different fixed point in the resulting divisor; if we call these points P and Q then we recover precisely the above linear equivalence (1), and hence leads to the same Bäcklund transformation.

2. The Mumford system

2.1. Translations on hyperelliptic Jacobians

For a fixed integer $g \geq 1$ the phase space M_g of the (g-dimensional) Mumford system (see [16]) is the affine space M_g of Lax matrices $L(x)$ of the form

$$L(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix},$$

where $u(x)$, $v(x)$ and $w(x)$ are polynomials, subject to the following constraints: $u(x)$ and $w(x)$ are monic and their degrees are respectively g and $g+1$; the degree of $v(x)$ is at most $g-1$. Writing

$$u(x) = x^g + u_1x^{g-1} + \cdots + u_g, \qquad v(x) = v_1x^{g-1} + \cdots + v_g,$$
$$w(x) = x^{g+1} + w_0x^g + \cdots + w_g,$$

we can take the coefficients of these three polynomials as coordinates on M_g. In particular we will sometimes denote points of M_g by triples $(u(x), v(x), w(x))$. Let us denote by $\mathcal{P}_n$ the affine space of polynomials $f \in \mathbb{C}[x]$ which are monic and have degree n. We will usually view $\mathcal{P}_{2g+1}$ (or, in the next section, $\mathcal{P}_{2g+2}$) as the space of hyperelliptic curves with equation $y^2 = f(x)$; when all roots of f are distinct then such a curve is smooth and its genus is g. We denote such an affine curve by Γ_f and denote its smooth compactification, which is a compact Riemann surface, by $\bar{\Gamma}_f$. It is well known that every compact hyperelliptic Riemann surface of genus g is obtained in this way. The surjective map $\chi : M_g \to \mathcal{P}_{2g+1}$ defined by

$$\chi(L(x)) = -\det L(x) = u(x)w(x) + v^2(x) \tag{4}$$

is the moment map of an algebraic completely integrable system (a.c.i. system). This means in the first place that there is a Poisson structure[2] on M_g with respect to which $\chi^*(\mathcal{O}(\mathcal{P}_{2g+1}))$ is involutive (commutative for the Poisson bracket). Secondly, it means that the tangent space to a generic fiber $\chi^{-1}(f)$ of χ is spanned by the Hamiltonian vector fields associated to this involutive algebra; by the first condition these vector fields commute. Third, a generic fiber of χ is an affine part of a commutative algebraic group; in the present case, when the roots of f are distinct then $\chi^{-1}(f)$ is an affine part of a complex algebraic torus, namely it is isomorphic to the Jacobian of $\bar{\Gamma}_f$, minus its theta divisor. Finally, it means that the flow of the commuting Hamiltonian vector fields on each complex torus lifts to a linear flow on its universal covering space $\mathbb{C}^g$.

It is convenient for our constructions to introduce the universal curve $\mathcal{C}_g$ of $\mathcal{P}_{2g+1}$. Intuitively speaking, $\mathcal{C}_g$ is constructed out of $\mathcal{P}_{2g+1}$ by replacing every point of $\mathcal{P}_{2g+1}$ by the curve which it represents. Explicitly, $\mathcal{C}_g$ can be represented as the affine variety

$$\{(x, y, f) | x, y \in \mathbb{C},\ f \in \mathcal{P}_{2g+1} \text{ and } y^2 = f(x)\};$$

the natural projection $\mathcal{C}_g \to \mathcal{P}_{2g+1}$ will be denoted by π. The partial compactification of $\pi : \mathcal{C}_g \to \mathcal{P}_{2g+1}$, which is the quasi-projective variety obtained by compactifying the fibers of π, will be denoted as $\bar{\mathcal{C}}_g$ and we use the same notation π for the extension of π to $\bar{\mathcal{C}}_g$.

The first useful observation that we make is that any section ξ of $\pi : \mathcal{C}_g \to \mathcal{P}_{2g+1}$ leads to a family of transformations of phase space, where each transformation restricts to a translation on every Jacobian of the system. This follows from the fact that there is a natural section ξ_∞ of $\pi : \bar{\mathcal{C}}_g \to \mathcal{P}_{2g+1}$, which is given by $\xi_\infty(f) = (\infty_f, f)$, where ∞_f is the unique point needed to compactify Γ_f into $\bar{\Gamma}_f$. Indeed, if ξ is a section of $\pi : \Gamma_g \to \mathcal{P}_{2g+1}$

[2] There are in fact in the present case many (compatible) Poisson structures which make the Mumford system into an a.c.i. system, see [19] and Section 2.2.

then we get a commutative diagram

$$\begin{tikzcd} \bar{\mathcal{C}}_g & M_g \arrow[l, "\rho"'] \arrow[dl, "\chi"] \\ \mathcal{P}_{2g+1} \arrow[u, "\xi"] & \end{tikzcd}$$

where ρ is defined as $\rho = \xi \circ \chi$ and we get a map $B_\xi : M_g \to M_g$ by

$$\mathcal{L} \mapsto \mathcal{L} \otimes [\rho(\mathcal{L}) - \rho_\infty(\mathcal{L})], \tag{5}$$

where $\rho_\infty = \xi_\infty \circ \chi$. In this definition we use the fact that a generic point $L(x)$ of M_g (more precisely: each point of any fiber $\chi^{-1}(f)$ for which Γ_f is smooth) admits a natural interpretation as a holomorphic line bundle $\mathcal{L}$ of degree g over the Riemann surface $\bar{\Gamma}_f$, where $f = \chi(L(x))$; thus $\mathcal{L} \in \mathrm{Pic}^g(\bar{\Gamma}_f) \cong \mathrm{Jac}(\Gamma_f)$. Also, $[D]$ stands for the line bundle associated to a divisor D. By construction, the restriction of B_ξ to a generic level $\chi^{-1}(f)$ of the moment map χ is a translation over $[\xi(f) - \xi_\infty(f)]$. On the one hand, this implies that B_ξ is isospectral: it leaves the fibers of χ invariant. On the other hand, translations in a commutative group obviously preserve translation invariant vector fields, hence B_ξ leaves invariant all those vector fields on M_g which restrict to translation invariant vector fields on a generic fiber of χ; in particular each B_ξ leaves the integrable vector fields of the Mumford system invariant. Notice that it is unavoidable for such translation maps to have poles, because a non-zero translation moves the theta divisor, hence every fiber of χ will have a divisor of points which are sent out of phase space.

Our second observation is that the maps B_ξ can be effectively computed. Indeed, following Mumford (who attributes this construction to Jacobi) the above mentioned interpretation of a generic element $L(x) \in M_g$ as a line bundle $\mathcal{L}$ can be carried out explicitly as follows: to the point $L(x) = (u(x), v(x), w(x)) \in \chi^{-1}(f)$ we associate the divisor $D = \sum_{i=1}^{g}(x_i, y_i)$ on Γ_f (hence the line bundle $\mathcal{L} = [D]$ on $\bar{\Gamma}_f$, when f is supposed to have no multiple roots) using the following simple prescription:

$$x_1, \ldots, x_g \text{ are the zeros of } u(x), \tag{6}$$

$$y_i = v(x_i) \quad \text{for } i = 1, \ldots, g. \tag{7}$$

Assuming $(u(x), v(x), w(x))$ to be generic, we let $\tilde{L}(x) = B_\xi(L(x))$ which we also write as

$$(\tilde{u}(x), \tilde{v}(x), \tilde{w}(x)) = B_\xi(u(x), v(x), w(x)).$$

Since $(u(x), v(x), w(x))$ is generic its image does indeed belong to M_g. We denote by D the divisor $\sum_{i=1}^{g}(x_i, y_i)$ given by (6) and (7). According to (5) the line bundle to which $[D]$ is mapped is obtained by tensoring with $[\rho[D] - \rho_\infty[D]]$. We define regular functions λ and μ on $\mathcal{P}_{2g+1}$ by $\xi(f) = (\lambda(f), \mu(f), f)$; in order to simplify the notation we will write λ_f and μ_f for $\lambda(f)$ and $\mu(f)$. Then (6) and (7) associate to $(\tilde{u}(x), \tilde{v}(x), \tilde{w}(x))$ the line bundle $\tilde{\mathcal{L}} = [\tilde{D}]$ for which we have two different descriptions,

$$[\tilde{D}] = \left[\sum_{i=1}^{g}(\tilde{x}_i, \tilde{y}_i)\right] = \left[\sum_{i=1}^{g}(x_i, y_i) + (\lambda_f, \mu_f) - \infty_f\right].$$

The second equality expresses that $\sum_{i=1}^{g}(\tilde{x}_i, \tilde{y}_i) + \infty_f$ and $\sum_{i=1}^{g}(x_i, y_i) + (\mu_f, \lambda_f)$ are linearly equivalent. This means that there is a rational function (unique up to a non-zero constant) on $\bar{\Gamma}$ with poles at (x_i, y_i) $(i = 1, \dots, g)$ and (λ_f, μ_f) and with a zero at ∞_f. For any $\beta \in \mathbb{C}$ we consider

$$F(x, y) = \frac{y + v(x) + \beta u(x)}{u(x)(x - \lambda_f)}. \tag{8}$$

Taking a local parameter t at ∞_f, such as $x = 1/t^2$ and $y = 1/t^{2g+1}(1 + \mathrm{O}(t))$, we find that F has a zero at ∞_f. Moreover, both the numerator and denominator vanish at the points $(x_i, -y_i)$, hence it is sufficient to have that β is such that the numerator vanishes at $(\lambda_f, -\mu_f)$ to have the required function. Thus we take β to be given by

$$\beta = \frac{\mu_f - v(\lambda_f)}{u(\lambda_f)} = \frac{w(\lambda_f)}{\mu_f + v(\lambda_f)}. \tag{9}$$

Notice that β depends on the phase variables; one may think of β itself as being a phase variable, depending on the other phase variables (see also Section 2.3). The zeros of F on $\bar{\Gamma}_f$ are the points $(\tilde{x}_i, \tilde{y}_i)$ and cannot be explicitly computed as such. However, the polynomials $(\tilde{u}(x), \tilde{v}(x), \tilde{w}(x))$ to which they correspond, take a simple form. Consider

$$(y - v(x) - \beta u(x))F(x, y) = \frac{y^2 - (v(x) + \beta u(x))^2}{u(x)(x - \lambda_f)} = \frac{w(x) - 2\beta v(x) - \beta^2 u(x)}{x - \lambda_f}.$$

Counting degrees we find that the last expression is monic of degree g in x and is independent of y, hence it is $\prod_{i=1}^{g}(x - \tilde{x}_i)$, i.e. it is $\tilde{u}(x)$. Thus we have obtained an explicit expression for the first component of B_ξ as follows:

$$\tilde{u}(x) = \frac{\beta^2 u(x) + 2\beta v(x) - w(x)}{\lambda_f - x}. \tag{10}$$

We claim that the second component of B_ξ is given by

$$\begin{aligned}\tilde{v}(x) &= -v(x) - \beta u(x) + \beta \tilde{u}(x) \\ &= \frac{\beta(x - \lambda_f + \beta^2)u(x) + (x - \lambda_f + 2\beta^2)v(x) - \beta w(x)}{\lambda_f - x}.\end{aligned} \tag{11}$$

To show this, it suffices to verify that for generic $(u(x), v(x), w(x))$ both sides take the same value on g different points (both sides are of degree at most $g - 1$ in x). This is easily done by using the points $(\tilde{x}_j, \tilde{y}_j)$ $(j = 1, \dots, g)$; just express that $(\tilde{x}_j, \tilde{y}_j) \in \Gamma_f$ and $F(\tilde{x}_j, \tilde{y}_j) = 0$ for $1 \le j \le g$, to find that

$$\tilde{y}_j = \tilde{v}(\tilde{x}_j) = -v(\tilde{x}_j) - \beta u(\tilde{x}_j)$$

for $j = 1, \dots, g$. The formula for $\tilde{w}(x)$ follows from

$$\tilde{u}(x)\tilde{w}(x) + \tilde{v}^2(x) = f(x) = u(x)w(x) + v^2(x),$$

giving

$$\tilde{w}(x) = -\frac{(x-\lambda_f+\beta^2)^2u(x)+2\beta(x-\lambda_f+\beta^2)v(x)-\beta^2w(x)}{\lambda_f - x}. \tag{12}$$

Eqs. (10)–(12) give explicit formulas for all maps B_ξ (ξ any section of $\mathcal{C}_g \to \mathcal{P}_{2g+1}$). We will investigate the Poissonicity of the maps B_ξ in Section 2.2.

We finish this section by rewriting B_ξ in terms of matrices. Since B_ξ preserves by construction the spectrum of the Lax matrix $L(x)$, it must be given by a similarity transformation of $L(x)$,

$$\tilde{L}(x) = M(x)L(x)M(x)^{-1}. \tag{13}$$

It is easy to verify that such a matrix M is given by the formula

$$M(x) = \begin{pmatrix} \beta & x-\lambda_f+\beta^2 \\ 1 & \beta \end{pmatrix}. \tag{14}$$

Notice that $\det M(x) = \lambda_f - x$.

2.2. Poissonicity

There are many (compatible) Poisson structures for the Mumford system on M_g and they can be obtained from a reduction of a natural class of R brackets on the loop algebra of $\mathfrak{sl}(2)$ (see [19]) or from (almost) canonical brackets on the linearizing variables (see [22]). Explicitly, there is a Poisson structure for any univariate polynomial $\varphi(x)$ of degree at most g and they are given by the following Poisson brackets for the polynomials $u(x)$, $v(x)$ and $w(x)$:

$$\begin{aligned}
&\{u(x),u(y)\}^\varphi = \{v(x),v(y)\}^\varphi = 0, \qquad \{u(x),v(y)\}^\varphi = \frac{u(x)\varphi(y)-u(y)\varphi(x)}{x-y},\\
&\{u(x),w(y)\}^\varphi = -2\frac{v(x)\varphi(y)-v(y)\varphi(x)}{x-y},\\
&\{v(x),w(y)\}^\varphi = \frac{w(x)\varphi(y)-w(y)\varphi(x)}{x-y} - u(x)\varphi(y),\\
&\{w(x),w(y)\}^\varphi = 2(v(x)\varphi(y)-v(y)\varphi(x)).
\end{aligned} \tag{15}$$

We will show that $B_\xi : (u(x), v(x), w(x)) \to (\tilde{u}(x), \tilde{v}(x), \tilde{w}(x))t$ is a Poisson map for those sections ξ for which λ depends on the Casimirs of $\{\cdot,\cdot\}^\varphi$ only. More precisely, denoting the algebra of Casimirs of $\{\cdot,\cdot\}^\varphi$ by Z^φ we assume in the sequel that λ factors over the canonical[3] map $p : \mathcal{P}_{2g+1} \to \mathrm{Spec}\, Z^\varphi$, as in the following diagram:

$$\begin{tikzcd}
 & \mathbb{C} \\
\mathrm{Spec}\, Z^\varphi \arrow[ur] & \mathcal{P}_{2g+1} \arrow[l, "p"'] \arrow[u, "\lambda"']
\end{tikzcd}$$

[3] p is dual to the algebra homomorphism $Z^\varphi \hookrightarrow \mathcal{O}(\mathcal{P}_{2g+1})$.

This assumption implies that λ has trivial brackets with all phase variables; notice that this does not imply that μ has trivial brackets with all phase variables. One particular case of interest is when λ is constant.

Using (15) it can be shown by direct computation that the Poisson brackets of the tilded variables are the same as those of the untilded variables—which proves that B_ξ is a Poisson map—but such computations are very long and tedious. However, by using the Poisson bracket formalism that was introduced by the Leningrad school these computations become feasible. In this formalism one computes the 4×4 matrix $\{L(x) \overset{\otimes}{,} L(y)\}$, which is defined similarly as the tensor product of $L(x)$ and $L(y)$, but taking the Poisson bracket of entries of $L(x)$ with entries of $L(y)$ instead of their product. Using this notation (15) can be written as

$$\begin{aligned}\{L(x) \overset{\otimes}{,} L(y)\} = &[r(x-y), L_1(x)\varphi(y) + \varphi(x)L_2(y)] \\ &- [\sigma \otimes \sigma, L_1(x)\varphi(y) - \varphi(x)L_2(y)],\end{aligned} \tag{16}$$

where $L_1(x) = L(x) \otimes \mathrm{Id}$, $L_2(y) = \mathrm{Id} \otimes L(y)$,

$$\sigma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$r(x) = -\frac{1}{x}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We need to verify that (16) also holds for the tilded variables, which means, using $\tilde{L}(x) = M(x)L(x)M(x)^{-1}$, that

$$\begin{aligned}&\{M(x)L(x)M(x)^{-1} \overset{\otimes}{,} M(y)L(y)M(y)^{-1}\} \\ &\quad = [r(x-y), M(x)L(x)M(x)^{-1} \otimes \mathrm{Id}\varphi(y) + \varphi(x)\mathrm{Id} \otimes M(y)L(y)M(y)^{-1}] \\ &\qquad - [\sigma \otimes \sigma, M(x)L(x)M(x)^{-1}\varphi(y) \otimes \mathrm{Id} - \mathrm{Id} \otimes \varphi(x)M(y)L(y)M(y)^{-1}].\end{aligned} \tag{17}$$

In order to compute the left-hand side of this equation we need explicit formulas for $\{L(x) \overset{\otimes}{,} M(y)\}$, for $\{M(x) \overset{\otimes}{,} L(y)\}$ and for $\{M(x) \overset{\otimes}{,} M(y)\}$. It is easy to see that $\{M(x) \overset{\otimes}{,} M(y)\} = 0$. In order to find the other brackets we need the brackets of β with the other phase variables. They were computed from the definition (9) of β, using the identity $\{\mu_f^2 - u(\lambda_f)w(\lambda_f) - v^2(\lambda_f), \cdot\}^\varphi = 0$.

$$\{u(x), \beta\}^\varphi = \frac{\mu_f\varphi(x) - \varphi(\lambda_f)(v(x) + \beta u(x))}{\mu_f(x-\lambda_f)},$$

$$\{v(x), \beta\}^\varphi = -\frac{2\mu_f\beta\varphi(x) - \varphi(\lambda_f)(\beta^2 u(x) + w(x) - u(x)(x-\lambda_f))}{2\mu_f(x-\lambda_f)},$$

$$\{w(x), \beta\}^\varphi = -\frac{(\beta^2 + x - \lambda_f)\mu_f\varphi(x) + \varphi(\lambda_f)(\beta^2 v(x) - \beta w(x) - v(x)(x-\lambda_f))}{\mu_f(x-\lambda_f)}.$$

Using these formulas it is easy to verify that

$$\{L(x)\underset{,}{\otimes}M(y)\}=\left(\frac{\varphi(\lambda_f)}{2\mu_f}\left[L(x),M(x)^{-1}\frac{\partial M}{\partial\beta}\right]+\varphi(x)M(x)^{-1}\epsilon\right)\otimes\frac{\partial M}{\partial\beta},$$
$$\{M(x)\underset{,}{\otimes}L(y)\}=-\frac{\partial M}{\partial\beta}\otimes\left(\frac{\varphi(\lambda_f)}{2\mu_f}\left[L(y),M(y)^{-1}\frac{\partial M}{\partial\beta}\right]+\varphi(y)M(y)^{-1}\epsilon\right),$$

where $\epsilon=\mathrm{diag}(1,-1)$. For future use we note the following identity:

$$\begin{aligned}&\mathrm{Ad}_{M(x)\otimes M(y)}(r(x-y)+\sigma\otimes\sigma)\\&\quad=r(x-y)+\sigma\otimes\sigma-\epsilon M(x)^{-1}\otimes\frac{\partial M}{\partial\beta}M(y)^{-1}.\end{aligned}\tag{18}$$

Since $\{M(x)\underset{,}{\otimes}M(y)\}=0$ we get

$$\begin{aligned}&\{M(x)L(x)M(x)^{-1}\underset{,}{\otimes}M(y)L(y)M(y)^{-1}\}\\&\quad=\mathrm{Id}\otimes M(y)\{M(x)\underset{,}{\otimes}L(y)\}L(x)M(x)^{-1}\otimes M(y)^{-1}\\&\qquad+M(x)\otimes\mathrm{Id}\{L(x)\underset{,}{\otimes}M(y)\}M(x)^{-1}\otimes L(y)M(y)^{-1}\\&\qquad+M(x)\otimes M(y)\{L(x)\underset{,}{\otimes}L(y)\}M(x)^{-1}\otimes M(y)^{-1}\\&\qquad-M(x)\otimes M(y)L(y)M(y)^{-1}\{L(x)\underset{,}{\otimes}M(y)\}M(x)^{-1}\otimes M(y)^{-1}\\&\qquad-M(x)L(x)M(x)^{-1}\otimes M(y)\{M(x)\underset{,}{\otimes}L(y)\}M(x)^{-1}\otimes M(y)^{-1}.\end{aligned}$$

From here on the computation is straightforward: substitute the above expressions for $\{L(x)\underset{,}{\otimes}L(y)\}$, $\{L(x)\underset{,}{\otimes}M(y)\}$ and $\{M(x)\underset{,}{\otimes}L(y)\}$ and use, besides the identity (18) the following formulas, valid for arbitrary matrices: $(A\otimes B)(C\otimes D)=AC\otimes BD$ and $[A\otimes B,C\otimes D]=AC\otimes BD-CA\otimes DB$. Notice that since each expression is either linear in $\varphi(\lambda_f)$, in $\varphi(x)$ or in $\varphi(y)$ the computation can be split up in three shorter verifications.

It follows that $B_\xi:(u(x),v(x),w(x))\to(\tilde u(x),\tilde v(x),\tilde w(x))$ is a Poisson map for those sections ξ for which λ depends on the Casimirs of $\{\cdot,\cdot\}^\varphi$ only. In view of the preceding section they are Bäcklund transformations.

2.3. *The existence of a section ξ*

We have deliberately omitted the question of the *existence* of a (global) section ξ of $\pi:\mathcal{C}_g\to\mathcal{P}_{2g+1}$. In fact it is easy to show that in the case of the Mumford system such a (global) section does not exist. Indeed, let us suppose that $\lambda:\mathcal{P}_{2g+1}\to\mathbb{C}$ is given. Since $\mathcal{P}_{2g+1}$ consists of all monic polynomials of degree $2g+1$ ($g\geq 1$) the regular function $f\mapsto f(\lambda_f)$, defined on $\mathcal{P}_{2g+1}$, is never a constant map. Therefore it takes the value 0 at some point f_0, without being identically zero on any neighborhood of f_0. If λ is to be the first component of a section ξ, i.e. $\xi(f)=(\lambda_f,\mu_f,f)$ then μ_f must be a regular map on the affine space $\mathcal{P}_{2g+1}$, satisfying $\mu_f^2=f(\lambda_f)$. On any neighborhood of f_0 this is however impossible. On the other hand, it is clear that in a small neighborhood U of any $f\in\mathcal{P}_{2g+1}$ a section ξ exists: choose $\lambda:\mathcal{P}_{2g+1}\to\mathbb{C}$ such that $f(\lambda_f)\neq 0$. Thus the constructed Bäcklund transformations should either be interpreted semi-locally (i.e. on a neighborhood

$\chi^{-1}(U)$, where U is a neighborhood of a fixed $f_0 \in \mathcal{P}_{2g+1}$), or one has to think of the Bäcklund transformation B_ξ as a two-valued map. In the latter interpretation it is worth to observe that the two translations which one obtains are opposite to each other, as follows from

$$[(x, y) + (x, -y) - 2\infty_f] = 0,$$

valid for any $(x, y) \in \Gamma_f$. On the one hand, this implies that in a sense B_ξ is its own inverse, on the other hand, it implies that even an n-fold iteration of B_ξ is only 2-valued, not 2^n-valued.

If one insists on having a Bäcklund transformation which is single-valued then one has to pass to a cover of phase space, precisely as in the classical construction of Riemann surfaces as the natural objects on which multi-valued algebraic functions become single-valued. We wish to show now that this larger phase space inherits in fact a Poisson structure and an a.c.i. system from the Mumford system, so that we have, in fact, constructed a single-valued map for an a.c.i. system, which reduces to the Mumford system after taking the quotient by an involution. Our arguments will be given here for the Mumford system, but apply also to other systems, the involution being in general replaced by a higher order automorphism. We fix a regular map $\lambda : \mathcal{P}_{2g+1} \to \mathbb{C}$ and define the following quasi-projective variety:

$$M_g^\lambda = \{(u, v, w, \beta)|(u, v, w) \in M_g,\ (\beta u(\lambda_f) + v(\lambda_f))^2 = f(\lambda_f),\ u(\lambda_f) \neq 0\}.$$

The natural map $M_g^\lambda \to M_g$ is a two-fold ramified cover, and the dynamics on this larger space, in particular the Poisson brackets of u, v and w with β follow from the relation

$$\{(\beta u(\lambda_f) + v(\lambda_f))^2 - f(\lambda_f), \cdot\} = 0$$

(see [21] for general constructions of this type). Since all our formulas for the Bäcklund transformation were expressed regularly in terms of u, v, w and β only, the Bäcklund transformation is single-valued on this larger space. Obviously, the functions in involution of the Mumford system lead to an algebra of functions in involution on the cover and, since the dimension did not change, they still form an integrable system. To show that it is actually an a.c.i. system we must investigate the nature of the generic fiber of the moment map. For a generic $f \in \mathcal{P}_{2g+1}$ we have that $f(\lambda_f) \neq 0$. If we denote the two square roots of $f(\lambda_f)$ by $\pm\mu_f$ then the fiber over f is reducible and its two components are given by

$$u(x)w(x) + v^2(x) = f(x), \qquad \beta u(\lambda_f) + v(\lambda_f) = \pm\mu_f.$$

Notice that the two components do not intersect. Since we know that the variety in M_g, given by $u(x)w(x) + v^2(x) = f(x)$ is an affine part of the Jacobian $\mathrm{Jac}(\bar{\Gamma}_f)$, we find that each component is an affine part of $\mathrm{Jac}(\bar{\Gamma}_f)$; due to the fact that $u(\lambda_f) = 0$ along some divisor, the divisor which is removed in the latter case is slightly larger than the one removed in the former case. Since the lifted vector fields are also linear on these Jacobians this shows that the integrable system that we have constructed is actually an a.c.i. system (with reducible fibers).

Another way in which a global section ξ in the case of the Mumford system can be found is by passing to a subsystem, i.e. restricting phase space and its Poisson structure to a hyperplane on which the algebra of functions in involution restricts to an a.c.i. system. This smaller a.c.i. system is also universal for hyperelliptic curves in the sense that, just as for the Mumford system, every hyperelliptic Jacobian (minus its theta divisor) appears as one of the fibers of its moment map. Suppose that $\mathcal{F}$ is an affine subspace of $\mathcal{P}_{2g+1}$ and λ is a regular (or rational) function on $\mathcal{F}$ such that the $f(\lambda_f) = c$, where c is a constant, $c \in \mathbb{C}$. It can be shown that this implies that the map λ is constant. By adding $-c$ to all elements of f we find that all these polynomials have a common root r. By replacing $x \to x + r$ in $f(x)$ this amounts to saying that up to isomorphism the only reasonable subvariety of M_g on which a global section ξ can exist is the subspace[4] M'_g of polynomials $(u(x), v(x), w(x))$ for which $u(0)w(0) + v^2(0) = 0$; the map λ must then be the zero map, the section is given by $\lambda_f = (0, 0, f)$ and the translation on every fiber is given by $[(0, 0)_f - \infty_f]$. Then $\beta = -v_g/u_g = w_g/v_g$ and the Bäcklund transformation takes the following form:

$$\tilde{u}_i = w_{i-1} - 2\frac{w_g v_{i-1}}{v_g} + \frac{w_g u_{i-1}}{u_g},$$
$$\tilde{v}_i = -v_i + \frac{v_g}{u_g}u_i - \frac{v_g w_{i-1}}{u_g} + 2\frac{w_g v_{i-1}}{u_g} - \frac{v_g w_g u_{i-1}}{u_g^2}.$$

Since $(0, 0)_f$ is a Weierstraß point for any $f \in \mathcal{F}$ the divisor $2((0, 0)_f - \infty_f)$ is linearly equivalent to zero, in other words $(0, 0)_f - \infty_f$ is a half period (two-torsion point) on each Jacobian. This explains why the two opposite translations are identical and it shows that this Bäcklund transformation is an involution.[5]

2.4. Discretizations and continuum limits

We now wish to show that the maps B_ξ provide a discretization of the Mumford system. Mumford constructs for every element of $\mathbb{P}^1$ a vector field on M_g which is translation invariant (linear) when restricted to each fiber of χ. His vector field corresponding to ∞ is reconstructed here as the limit

$$\lim_{t\to 0} \frac{B_{\xi_t}(u(x), v(x), w(x)) - (u(x), v(x), w(x))}{t},$$

where $\xi_t : \mathcal{P}_{2g+1} \to \mathcal{C}_g$ converges as $t \to 0$ to the constant section $\xi_\infty : \mathcal{P}_{2g+1} \to \bar{\mathcal{C}}_g : f \mapsto \infty_f$. The limit taken here is the one for which the sections $\xi_t(f) = (\lambda_f(t), \mu_f(t), f)$ take the form

$$\xi_t(f) = \left(\frac{1}{t^2}, \frac{1}{t^{2g+1}}\left(1 + \frac{a_0}{2}t^2 + \mathrm{O}(t^4)\right), f\right), \tag{19}$$

where $a_0 = u_1 + w_0$ is the second coefficient of f, i.e. $f(x) = x^{2g+1} + a_0 x^{2g} + \cdots$. Then

$$\beta = \frac{1}{t}\left(1 + \frac{w_0 - u_1}{2}t^2 + \mathrm{O}(t^3)\right),$$

[4] This happens to be a Poisson subspace for many (but not all) of the Poisson structures on M_g, see [19] or Section 2.2.

[5] The fact that this Bäcklund transformation is an involution should not be confused with our earlier claim that *in a sense* the Bäcklund transformation is its own inverse.

hence (10)–(12) take the form

$$\begin{aligned}\tilde{u}(x) &= u(x) + 2tv(x) + \mathrm{O}(t^2),\\ \tilde{v}(x) &= v(x) - t(w(x) - (x - u_1 + w_0)u(x)) + \mathrm{O}(t^2),\\ \tilde{w}(x) &= w(x) - 2t(x - u_1 + w_0)v(x) + \mathrm{O}(t^2).\end{aligned} \tag{20}$$

The coefficient of t in (20) is (up to a factor of 2) precisely Mumford's vector field X_∞ (see [16, p. 3.43]).

Let us now turn to Mumford's general vector fields X_a ($a \in \mathbb{P}^1$). These vector fields have the property of being tangent to the curves $P \mapsto [P + (g-1)\infty]$ at the points $(a, \pm b_f)$ on every curve f (here $b_f^2 = f(a)$), which suggests that these more general vector fields may be constructed by taking an appropriate limit $(\lambda_f, \mu_f) \to (a, b_f)$ of the composition of two Bäcklund transformations corresponding to a shift

$$[(\lambda_f, \mu_f) - (a, b_f)] = [(\lambda_f, \mu_f) + (a, -b_f) - 2\infty_f]$$

on each Jacobian. Our vector fields will be more general than Mumford's vector fields because we allow a_f to depend on f. Concretely, we will first shift over $[(a_f, -b_f) - \infty_f]$ and then over $[(\lambda_f(t), \mu_f(t)) - \infty_f]$; the matrices going with these transformations (as in (14)) will be denoted by $P(x)$ and $Q_t(x)$. Then

$$P(x) = \begin{pmatrix} -\beta & x - a_f + \beta^2 \\ 1 & -\beta \end{pmatrix}$$

with

$$\beta = \frac{b_f + v(a_f)}{u(a_f)} = \frac{w(a_f)}{b_f - v(a_f)}, \tag{21}$$

the transformed L is denoted by $\tilde{L}$ as in (13). In particular,

$$\tilde{u}(x) = \frac{w(x) + 2\beta v(x) - \beta^2 u(x)}{x - a_f}, \qquad \tilde{v}(x) = -v(x) + \beta u(x) - \beta \tilde{u}(x). \tag{22}$$

Also,

$$Q_t(x) = \begin{pmatrix} \beta(t) & x - \lambda_f(t) + \beta^2(t) \\ 1 & \beta(t) \end{pmatrix}$$

with

$$\beta(t) = \frac{\mu_f(t) - \tilde{v}(\lambda_f(t))}{\tilde{u}(\lambda_f(t))}.$$

Notice that $\beta(0) = \beta$ since $(\lambda_f(0), \mu_f(0)) = (a_f, b_f)$. Let $M_t(x) = Q_t(x)P(x)$ be the matrix defining their composition. To the deformation family $\tilde{L}_t(x) = M_t(x)L(x)M_t^{-1}(x)$ there corresponds a vector field on M_g, defined by

$$\frac{\mathrm{d}L}{\mathrm{d}t_{a_f}}(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (M_t(x)L(x)M_t^{-1}(x)).$$

In terms of $Q(x)$ this vector field is given by (a prime denotes a derivative with respect to t)

$$\frac{\mathrm{d}L}{\mathrm{d}t_{a_f}}(x)=[M_0'(x)M_0(x)^{-1},L(x)]=[Q_0'(x)Q_0^{-1}(x),L(x)].$$

We consider the family of sections $\xi_t=(\lambda_f(t),\mu_f(t),f)$, where $\lambda_f(t)=a_f+t$ and $\mu_f(t)=\sqrt{f(a_f+t)}$. We will show below that

$$\beta'(0)=\frac{u(a_f)}{2b_f}. \tag{23}$$

Then

$$\begin{aligned}Q_0'(x)Q_0^{-1}(x)&=-\frac{1}{2b_f(x-a_f)}\begin{pmatrix}u(a_f)&2v(a_f)\\0&u(a_f)\end{pmatrix}\begin{pmatrix}\beta&a_f-x-\beta^2\\-1&\beta\end{pmatrix}\\&=\frac{1}{2b_f(x-a_f)}\begin{pmatrix}v(a_f)-b_f&w(a_f)+u(a_f)(x-a_f)\\u(a_f)&-v(a_f)-b_f\end{pmatrix}.\end{aligned}$$

Removing a diagonal matrix from this matrix we get the following Lax equations:

$$\frac{\mathrm{d}L}{\mathrm{d}t_{a_f}}(x)=\frac{1}{2b_f}\left[\frac{L(a_f)}{x-a_f}+\begin{pmatrix}0&u(a_f)\\0&0\end{pmatrix},L(x)\right],$$

which reduces, when $a_f=a$ is chosen independently of f, to Mumford's vector field X_a (up to a factor $2b_f$ which can be absorbed in t).

Formula (23) remains to be shown.

$$\begin{aligned}\beta'(0)&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\frac{\mu_f(t)-\tilde v(\lambda_f(t))}{\tilde u(\lambda_f(t))}\\&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\frac{\mu_f(t)+v(a_f+t)-\beta u(a_f+t)}{w(a_f+t)+2\beta v(a_f+t)-\beta^2u(a_f+t)}t\\&=\lim_{t\to 0}\frac{\mu_f(t)+v(a_f+t)-\beta u(a_f+t)}{w(a_f+t)+2\beta v(a_f+t)-\beta^2u(a_f+t)}\\&=\frac{\mu'_f(0)+v'(a_f)-\beta u'(a_f)}{w'(a_f)+2\beta v'(a_f)-\beta^2u'(a_f)}.\end{aligned}$$

Taking the derivative of $\mu_f^2(t)=u(\lambda_f(t))w(\lambda_f(t))+v^2(\lambda_f(t))$ at $t=0$ we obtain

$$\mu'_f(0)=\frac{1}{2b_f}(u(a_f)w'(a_f)+u'(a_f)w(a_f)+2v(a_f)v'(a_f)),$$

and $w(a_f)$ is easily eliminated from this equation by using $w(a_f)=-2\beta v(a_f)+\beta^2u(a_f)$, a consequence of (22). The announced formula for $\beta'(0)$ follows after substituting this value of $\mu'_f(0)$, upon using (21).

2.5. Normalizations of eigenvectors of Lax operators

In this section we describe another approach to Bäcklund transformations and we explain how the two approaches are related. For this approach we assume that the a.c.i. system is given in Lax form.

Let us recall (see, e.g. [8]) that a generic Lax matrix $L(x) \in \mathrm{End}(\mathbb{C}^{n+1})[x]$ defines a line bundle on the associated spectral curve $\Gamma : \det(L(x) - y\,\mathrm{Id}) = 0$; generic means here that the affine curve Γ is assumed smooth and that for generic $(x, y) \in \Gamma$ the eigenspace of $L(x)$ corresponding to the eigenvalue y is one-dimensional (both conditions are verified for the generic $L(x)$ of the Mumford system). Assuming $L(x)$ to be generic we denote, as before, by $\bar{\Gamma}$ the compact Riemann surface corresponding to Γ and we consider the eigenvector map $\kappa : \bar{\Gamma} \to \mathbb{P}^n$, which is defined, on the affine piece Γ, by

$$L(x)\kappa(x, y) = y\kappa(x, y).$$

An explicit description of κ on an affine piece of $\bar{\Gamma}$ is given by the map

$$\kappa_i : (x, y) \mapsto (L(x) - y\,\mathrm{Id})_i^{\wedge}, \tag{24}$$

where $1 \leq i \leq n + 1$ is arbitrary, $A^{\wedge}$ stands for the adjoint of the matrix A and A_i stands for the ith column of A. More precisely, every κ_i is defined on $\Gamma \setminus S_i$, where S_i is a collection of points and $\cap_i S_i = \emptyset$. We will see shortly that we need *all* local representatives κ_i $(i = 1, \ldots, n + 1)$ of κ for our computations. The line bundle $\mathcal{L}$, defined by $L(x)$, is given by $\mathcal{L} = \kappa^*\mathcal{H}$, where $\mathcal{H}$ is the hyperplane bundle on $\mathbb{P}^n$. The degree d of $\mathcal{L}$ follows from

$$\deg \mathcal{L} = \deg \kappa(\bar{\Gamma}) \deg \kappa. \tag{25}$$

It is a basic fact that pulling back a section s of $\mathcal{H}$ gives a section κ^*s whose zero locus is a divisor D on Γ such that $[D] = \kappa^*\mathcal{H}$ (see [9, Chapter 1.1]). Since a section of $\mathcal{H}$ is just a hyperplane, this gives us an explicit way to compute the line bundle $\mathcal{L} \in \mathrm{Pic}^d(\bar{\Gamma})$ from the Lax matrix:

$$\mathcal{L} = [\kappa^*(H \cap \kappa(\bar{\Gamma}))], \tag{26}$$

where H is any hyperplane in $\mathbb{P}^n$. Moreover, the isomorphism $\mathrm{Pic}^d(\bar{\Gamma}) \cong \mathrm{Jac}(\bar{\Gamma})$ is *not* canonical and depends on the choice of an element in $\mathrm{Pic}^{d-g}(\bar{\Gamma})$, a fact that we will now exploit to construct Bäcklund transformations.

To do this we assume that the given $L(x)$ is generic in the above sense; without loss of generality we may also assume that the image curve $\kappa(\bar{\Gamma})$ is non-degenerate (i.e. it is not contained in a hyperplane). Our main assumption, which will be relaxed in Section 3, is that $\deg \mathcal{L} = g + n$. Since the hyperplane bundle $\mathcal{H}$ on $\mathbb{P}^n$ is the line bundle which corresponds to any hyperplane of $\mathbb{P}^n$, fixing a section of $\mathcal{H}$ is equivalent to fixing a hyperplane H of $\mathbb{P}^n$. By non-degeneracy this can be done by fixing n points p_i on $\bar{\Gamma}$ which are in general position, and asking that H be such that $\sum p_i \leq \kappa^*H$ (when all p_i are different this means that $H = \mathrm{span}\{\kappa(p_i)\}$). Let us take another collection of n points $\tilde{p}_i$ in general position. We denote the corresponding hyperplane by $\tilde{H}$. If $\tilde{L}(x)$ is another Lax matrix, isospectral to $L(x)$, with corresponding map $\tilde{\kappa} : \bar{\Gamma} \to \mathbb{P}^n$ then we will say that $\tilde{L}(x) = B(L(x))$ if

$$\tilde{\kappa}^*(\tilde{H} \cap \tilde{\kappa}(\bar{\Gamma})) - \sum_{i=1}^{n} \tilde{p}_i = \kappa^*(H \cap \kappa(\bar{\Gamma})) - \sum_{i=1}^{n} p_i. \tag{27}$$

Notice that (27) implies that

$$\tilde{\mathcal{L}} = \mathcal{L} \otimes [\tilde{p}_1 - p_1] \otimes \cdots \otimes [\tilde{p}_n - p_n], \tag{28}$$

where $\mathcal{L}$ is given by (26) and $\tilde{\mathcal{L}}$ is defined analogously. One notices that this equation is the n-point analog of Eq. (5). In fact, let us specialize this to the case $n = 1$ and globalize the construction to the phase space of the Mumford system and recover exactly the Bäcklund transformations that we have constructed before.

If $L(x)$ is a generic matrix of M_g (the phase space of the Mumford system) then $n = 1$ and the two local representatives (24) of the eigenvalue map κ are given by

$$\kappa_1 : (x, y) \mapsto \begin{pmatrix} -v(x) - y \\ -u(x) \end{pmatrix} \quad \text{and} \quad \kappa_2 : (x, y) \mapsto \begin{pmatrix} -w(x) \\ v(x) - y \end{pmatrix}.$$

A hyperplane H of $\mathbb{P}$ is just a point: writing $\vec{\alpha} = (r : s)$ we find the following equations for the divisor $D = \kappa^*(H \cap \kappa(\Gamma_f))$:

$$0 = (v(x) + y)r + u(x)s, \qquad 0 = -w(x)r + (v(x) - y)s.$$

The degree of the image curve being 1 it suffices to determine the degree of κ to know the degree of the line bundle. Taking a $(r : s)$ generic, we easily find precisely $g + 1$ solutions hence $\deg \mathcal{L} = g+1$, showing that our main assumption is satisfied for the Mumford system. Since $n = 1$ we need to pick one point on every curve $\bar{\Gamma}_f$ to represent $\mathcal{L}$ as an element of the Jacobian $\mathrm{Jac}(\bar{\Gamma}_f) = \mathrm{Pic}^g(\bar{\Gamma}_f)$ and we need two points on every curve to construct a Bäcklund transformation as in (27). We do this by picking the sections ξ_∞ and ξ which were constructed in Section 2.1. For the first choice, which corresponds to picking the point ∞_f at every curve, we find $\vec{\alpha}_0 = (0 : 1)$; we let this choice correspond to the untilded variables. We let the second choice, which is given by $\xi(f) = (\lambda_f, \mu_f, f)$, correspond to the tilded variables and we find[6]

$$\vec{\alpha} = (\tilde{u}(\lambda_f) : -\tilde{v}(\lambda_f) - \mu_f) = (\tilde{v}(\lambda_f) - \mu_f : \tilde{w}(\lambda_f)).$$

In order to simplify the computation we will write $\vec{\alpha}$ as $(1 : -\beta)$; it will follow later that this definition of β agrees with the one given in (9). Eq. (28) now expresses that the solutions of

$$u(x) = 0, \qquad v(x) = y,$$

are the same as the solutions of

$$(1 - \beta) \begin{pmatrix} -\tilde{v}(x) - y & -\tilde{w}(x) \\ -\tilde{u}(x) & \tilde{v}(x) - y \end{pmatrix} = 0, \tag{29}$$

[6] Given $L(x)$ there are g $(g+1)$ values (λ, μ) where the first (second) representation breaks down, i.e. it may be of the form $\vec{\alpha} = (0 : 0)$. For generic $L(x)$ those two sets of values are disjoint, in the non-generic case it suffices to take a limit.

except that (29) also has (λ_f, μ_f) as a solution. If we eliminate y from (29) we find that $\tilde{w}(x) + 2\beta\tilde{v}(x) - \beta^2\tilde{u}(x) = 0$ has as solutions λ_f and the roots of u, so

$$u(x) = \frac{\beta^2\tilde{u}(x) - 2\beta\tilde{v}(x) - \tilde{w}(x)}{\lambda_f - x}. \tag{30}$$

In order to obtain the formula for $v(x)$ we take the first equation in (29), $-\tilde{v}(x) - y + \beta\tilde{u}(x) = 0$ which has among its roots the solutions of $u(x) = 0$ and $v(x) = y$. It follows that the same is true for the polynomial $-\tilde{v}(x) - v(x) - \beta u(x) + \beta\tilde{u}(x) = 0$, but since this polynomial has degree less than g it is zero, giving

$$v(x) = -\tilde{v}(x) - \beta u(x) + \beta\tilde{u}(x). \tag{31}$$

If we express that (λ_f, μ_f) is a solution to (29), then (31) implies

$$\beta = \frac{\tilde{v}(\lambda_f) + \mu_f}{\tilde{u}(\lambda_f)} = \frac{\mu_f - v(\lambda_f)}{u(\lambda_f)},$$

as in (9). It follows that formulas (30) and (31) describe exactly the maps B_ξ, given by (10) and (11), in their inverse form. Notice that we would have obtained an expression for the maps B_ξ in their direct form by expressing that the solutions to

$$\tilde{u}(x) = 0, \qquad \tilde{v}(x) = y,$$

are the same as the solutions of

$$(1-\beta)\begin{pmatrix} -v(x) - y & -w(x) \\ -u(x) & v(x) - y \end{pmatrix} = 0, \tag{32}$$

except that (32) also has $(\lambda_f, -\mu_f)$ as a solution (this follows from the linear equivalence $(\lambda_f, \mu_f) + (\lambda_f, -\mu_f) \sim_l 2\infty_f$).

It follows from [16] that the roots of the polynomial $u(x)$ lead to a separation of variables. This is one separation of variables; another one is given by Eq. (29) for the tilde variables. Relating them by assuming that they have the same divisor D as a solution, we create a Bäcklund transformation which corresponds to a shift on each Jacobian parameterized by a point (λ_f, μ_f) on its underlying curve Γ_f. Thus, in the Lax approach, our construction of Bäcklund transformations leads to alternative separation of variables (given one separation of variables) and given a pair of separations of variables we recover a Bäcklund transformation for the system.

2.6. Spectrality

We now come to a remarkable property of our Bäcklund transformations, which was baptized *spectrality* by [12]. In order to establish this property we will first consider an isomorphism to another integrable system in which the Poisson structure takes a simple form. We fix a monic polynomial $\varphi(x)$ of degree g and without multiple roots.

$$\varphi(x) = (x - a_1)(x - a_2)\cdots(x - a_g),$$

and we define an affine map $M_g \to \mathbb{C}^{3g+1}$ by

$$\frac{1}{\varphi(x)}(u(x), v(x), w(x)) = \left(1 + \sum_{i=1}^{g} \frac{f_i}{x - a_i}, \sum_{i=1}^{g} \frac{h_i}{x - a_i}, x + e_0 + \sum_{i=1}^{g} \frac{e_i}{x - a_i}\right).$$

Explicitly, the map can be computed in terms of the coordinates $e_0, \ldots, h_g$ on $\mathbb{C}^{3g+1}$ by

$$f_i = \frac{u(a_i)}{\prod_{k\neq i}(a_i - a_k)}, \qquad h_i = \frac{v(a_i)}{\prod_{k\neq i}(a_i - a_k)}, \qquad e_i = \frac{w(a_i)}{\prod_{k\neq i}(a_i - a_k)},$$

and $e_0 = w_0 - \sum_{i=1}^{g} a_i$. Dividing both sides of Eqs. (15) by $\varphi(x)\varphi(y)$ and taking residues at $x = a_i$ and $y = a_j$ we find that the variables $\{h_i, e_i, f_i\}_{i=1}^{g}$ are generators for the direct sum of g copies of the Lie–Poisson algebra of $\mathfrak{sl}(2)$: for $i, j = 1, \ldots, g$, we have $\{h_i, h_j\} = \{f_i, f_j\} = \{e_i, e_j\} = 0$ and

$$\{e_i, h_j\} = e_i\delta_{ij}, \qquad \{h_i, f_j\} = f_i\delta_{ij}, \qquad \{f_i, e_j\} = 2h_i\delta_{ij}. \tag{33}$$

Let us denote the Casimir element coming from the ith copy of $\mathfrak{sl}(2)$ by C_i, $C_i = h_i^2 + e_i f_i$. Then the equation of the spectral curve looks as follows:

$$\frac{y^2}{\varphi^2(x)} = \frac{f(x)}{\varphi^2(x)} = x + C_0 + \sum_{i=1}^{g}\left(\frac{C_i}{(x - a_i)^2} + \frac{H_i}{x - a_i}\right), \tag{34}$$

where

$$H_i = \sum_{j\neq i} \frac{2h_i h_j + e_i f_j + e_j f_i}{a_i - a_j} + e_i + (a_i + e_0) f_i,$$

and C_0 is an extra Casimir. If we define $\hat{\mu}_f = \mu_f/\varphi(\lambda_f)$ then

$$\hat{\mu}_f^2 = \lambda_f + C_0 + \sum_{i=1}^{g}\left(\frac{C_i}{(\lambda_f - a_i)^2} + \frac{H_i}{\lambda_f - a_i}\right),$$

and the relation (9) takes the form

$$\beta = \frac{\hat{\mu}_f - \sum_i h_i/(\lambda_f - a_i)}{1 + \sum_i f_i/(\lambda_f - a_i)}. \tag{35}$$

Notice that on $\mathbb{C}^{3g+1}$ the Poisson structure is independent of φ, but that the Hamiltonians are now dependent on the constants a_i which encode the Poisson structure $\{\cdot, \cdot\}^{\varphi}$ on M_g. In fact, the integrable system that we have obtained on $\mathbb{C}^{3g+1}$ is the first member of the deformed Gaudin magnet hierarchy from [7] and our Bäcklund transformations for the Mumford system are easily rewritten as Bäcklund transformations for this system. Explicitly we find

$$\tilde{f}_i = \frac{\beta^2 f_i + 2\beta h_i - e_i}{\lambda_f - a_i}, \qquad \tilde{h}_i = \frac{\beta(a_i - \lambda_f + \beta^2) f_i + (a_i - \lambda_f + 2\beta^2) h_i - \beta e_i}{\lambda_f - a_i},$$

$$\tilde{e}_i = -\frac{(a_i - \lambda_f + \beta^2)^2 f_i + 2\beta(a_i - \lambda_f + \beta^2) h_i - \beta^2 e_i}{\lambda_f - a_i}, \tag{36}$$

where β is given by (35).

We fix a section ξ of $\mathcal{C}_g \to \mathcal{P}_{2g+1}$ and we assume, as before, that λ_f depends on the Casimirs of $\{\cdot,\cdot\}^\varphi$ only, where $\xi(f) = (\lambda_f, \mu_f)$. We restrict our Bäcklund transformation B_ξ to a symplectic leaf of the Poisson structure by fixing generic values of all Casimirs C_j, $j = 0, \dots, g$. Then we have only $2g$ independent (Darboux-type) variables, which we choose to be $\{h_i, f_i\}_{i=1}^g$, we can express the e_i variables in terms of those (the expression for e_0 was computed from (34)),

$$e_i = \frac{C_i - h_i^2}{f_i}, \qquad e_0 = C_0 - \sum_{i=1}^g f_i,$$

and λ_f becomes a constant, so we drop the index f from the notation.

We will use the theory of canonical transformations to show that B_ξ has the spectrality property and we will find along the way an alternative, simpler, proof that B_ξ is a Poisson map. Recall that a transformation (bijective map) between ($2g$-dimensional) symplectic manifolds is canonical (symplectic) if and only if it has a local generating function F, i.e. in terms of canonical variables (x_i, y_i) and $(\tilde{x}_i, \tilde{y}_i)$ one has a function $F(x_1, \dots, x_g | \tilde{x}_1, \dots, \tilde{x}_g)$ such that

$$y_i = \frac{\partial F}{\partial x_i} \quad \text{and} \quad \tilde{y}_i = -\frac{\partial F}{\partial \tilde{x}_i}. \tag{37}$$

In turn this is equivalent to the compatibility relations

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}, \qquad \frac{\partial \tilde{y}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{y}_j}{\partial \tilde{x}_i}, \qquad \frac{\partial \tilde{y}_i}{\partial x_j} = -\frac{\partial y_j}{\partial \tilde{x}_i},$$

where $i, j = 1, \dots, g$; in these formulas one views the transformation locally as a map $(x_1, \dots, x_g, \tilde{x}_1, \dots, \tilde{x}_g) \to (y_1, \dots, y_g, \tilde{y}_1, \dots, \tilde{y}_g)$. In the present case this means that we have to view $h_1, \dots, h_g, \tilde{h}_1, \dots, \tilde{h}_g$ as functions of $f_1, \dots, f_g, \tilde{f}_1, \dots, \tilde{f}_g$ and that we need to verify the following compatibility relations:

$$f_j \frac{\partial h_i}{\partial f_j} = f_i \frac{\partial h_j}{\partial f_i}, \qquad \tilde{f}_j \frac{\partial \tilde{h}_i}{\partial \tilde{f}_j} = \tilde{f}_i \frac{\partial \tilde{h}_j}{\partial \tilde{f}_i}, \qquad f_j \frac{\partial \tilde{h}_i}{\partial f_j} = -\tilde{f}_i \frac{\partial h_j}{\partial \tilde{f}_i}. \tag{38}$$

To do this we need to express the variables $h_i, \tilde{h}_i$ and β in terms of the variables f_i and $\tilde{f}_i$. Multiplying both sides of (10) by $\lambda - x$ and comparing the leading terms in x we find $\beta^2 = \lambda + w_0 - u_1$, leading to the following expression for β as a function of $\{\tilde{f}_i, f_i\}_{i=1}^g$:

$$\beta^2 = \lambda + C_0 - \sum_{i=1}^g (\tilde{f}_i + f_i). \tag{39}$$

Excluding the e variables from Eq. (36) of the map $B_\xi : \{h_i, f_i\}_{i=1}^g \mapsto \{\tilde{h}_i, \tilde{f}_i\}_{i=1}^g$ we find the following $2g$ equations:

$$(h_i + \beta f_i)^2 - (\lambda - a_i)\tilde{f}_i f_i - C_i = 0, \tag{40}$$

$$\tilde{h}_i = -h_i + \beta(\tilde{f}_i - f_i). \tag{41}$$

Notice that with β from (39) the first equation defines h_i and then the second equation defines $\tilde{h}_i$, both as implicit functions of the variables $\{\tilde{f}_i, f_i\}_{i=1}^{g}$. Straightforward computation leads to

$$\frac{\partial h_i}{\partial f_j} = \frac{f_i}{2\beta} \quad \text{and} \quad \frac{\partial \tilde{h}_i}{\partial \tilde{f}_j} = -\frac{\tilde{f}_i}{2\beta}$$

for $i \neq j$ and to

$$\frac{\partial h_i}{\partial \tilde{f}_j} = \frac{f_i}{2\beta} + \frac{(\lambda - a_i) f_i}{2(h_i + \beta f_i)}\delta_{ij} \quad \text{and} \quad \frac{\partial \tilde{h}_i}{\partial f_j} = -\frac{\tilde{f}_i}{2\beta} - \frac{(\lambda - a_i)\tilde{f}_i}{2(h_i + \beta f_i)}\delta_{ij}$$

for any i, j. The compatibility conditions (38) follow at once.

In fact, in the same way we can prove another property of the Bäcklund transformation, its spectrality, which means that the variables $\hat{\mu}$ and λ are also canonical, in a sense, or more precisely, that the parameter λ enters in the generating function $F = F_\lambda$ in such a way that for the $\hat{\mu}$ being expressed in terms of $\{\tilde{f}_i, f_i\}_{i=1}^{g}$ variables we have a similar expression as in (37):

$$\hat{\mu} = \frac{\partial F_\lambda}{\partial \lambda}.$$

It follows that the following compatibility conditions are sufficient for proving the spectrality property of the Bäcklund transformation:

$$f_i \frac{\partial \hat{\mu}}{\partial f_i} = \frac{\partial h_i}{\partial \lambda} \quad \text{and} \quad \tilde{f}_i \frac{\partial \hat{\mu}}{\partial \tilde{f}_i} = -\frac{\partial \tilde{h}_i}{\partial \lambda}. \tag{42}$$

It is easily checked from (35) that these compatibility conditions indeed hold; the values of the two expressions in (42) are given by

$$-\frac{f_i}{2\beta} + \frac{f_i \tilde{f}_i}{2(h_i + \beta f_i)} \quad \text{and} \quad -\frac{\tilde{f}_i}{2\beta} + \frac{f_i \tilde{f}_i}{2(h_i + \beta f_i)}.$$

We have shown that our Bäcklund transformations are Poisson maps and have the spectrality property when φ is monic of degree g and has no multiple root. Obviously the fact that φ is monic is inessential. Moreover, all Poisson brackets are polynomial in terms of the roots a_i of φ hence these properties hold when φ is any polynomial of degree at most g.

2.7. Addition formulas for the $\wp$ function

In this section we show that our formulas (10) and (11) generalize the classical addition formulas for the Weierstraß $\wp$ function to the case of (families of) hyperelliptic curves. Let Γ be an elliptic curve, written in the Weierstraß form

$$Y^2 = 4X^3 - g_2 X - g_3.$$

Points on this curve are parameterized by $\wp$ and its derivative $\wp'$: for any $(X, Y) \in \Gamma$ there is a $z \in \mathbb{C}$ such that $(X, Y) = (\wp(z), \wp'(z))$. We write the equation of Γ as $y^2 =$

$f(x) = x^3 - (g_2/4)x - (g_3/4)$, thereby fixing $f \in \mathcal{P}_3$. We take two generic points on Γ and their sum (Γ is its own Jacobian, hence a group): $(\lambda_f, \mu_f) + (p, q) = (\tilde{p}, \tilde{q})$. On the one hand, we can associate to the points (p, q) and $(\tilde{p}, \tilde{q})$ the corresponding polynomials of the Mumford system, on the other hand, we can write them in terms of the $\wp$ function. As for the former we get

$$u(x) = x + u_1 = x - p, \qquad v(x) = v_1 = q,$$
$$w(x) = x^2 - u_1 x + w_1 = x^2 + px + \tfrac{1}{4}(4p^2 + g_2)$$

for (p, q) and we get similar formulas for $(\tilde{p}, \tilde{q})$ by putting tildes over all variables. In terms of $p, q, \tilde{p}$ and $\tilde{q}$ formulas (10), (11) and (9) (in that order) take the form

$$\beta^2 = p + \tilde{p} + \lambda, \qquad \beta = -\frac{q + \tilde{q}}{p - \tilde{p}} = \frac{\mu - q}{\lambda_f - p}. \tag{43}$$

As for the latter, let $(p, q) = (\wp(z), \wp'(z)/2)$, $(\tilde{p}, \tilde{q}) = (\wp(\tilde{z}), \wp'(\tilde{z})/2)$ and $(\lambda_f, \mu_f) = (\wp(z'), \wp'(z')/2)$. Then (43) reduces, after eliminating β to the following classical formulas:

$$\frac{1}{4}\left(\frac{\wp'(z) + \wp'(\tilde{z})}{\wp(z) - \wp(\tilde{z})}\right)^2 = \wp(z) + \wp(\tilde{z}) + \wp(z'),$$
$$\frac{1}{4}\left(\frac{\wp'(z') - \wp'(z)}{\wp(z') - \wp(z)}\right)^2 = \wp(z) + \wp(\tilde{z}) + \wp(z'),$$

where $\tilde{z} = z + z'$.

3. Bäcklund transformations in more complex situations

3.1. *The even Mumford system*

The Mumford system has a twin which was introduced by the second author in [23], where it was called the even master system; in this text we will call it the *even* Mumford system. The phase space M_g of the even Mumford system consists of Lax operators

$$L(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix},$$

where $u(x)$, $v(x)$ and $w(x)$ are now subject to the following constraints: $u(x)$ and $w(x)$ are monic and their degrees are respectively g and $g + 2$; the degree of $v(x)$ is at most $g - 1$. In this case we write

$$u(x) = x^g + u_1 x^{g-1} + \cdots + u_g, \qquad v(x) = v_1 x^{g-1} + \cdots + v_g,$$
$$w(x) = x^{g+2} + w_{-1} x^{g+1} + \cdots + w_g.$$

The map $\chi : M_g \to \mathcal{P}_{2g+1}$ is defined as in (4); notice that χ takes its values now in the affine space of monic polynomials of degree $2g + 2$, explaining the adjective *even*. The main difference between the even and the odd Mumford system is that the spectral curves $\Gamma_f : y^2 = f(x) = u(x)w(x) + v^2(x)$ have now two points at infinity, a fact which has drastic consequences for the geometry of the integrable system (see [21]).

Let us first construct Bäcklund transformations for this system by using the approach described in Section 2.1. We denote by $\mathcal{C}_g$ the universal curve over $\mathcal{P}_{2g+1}$ and we consider sections of the natural projection $\pi : \mathcal{C}_g \to \mathcal{P}_{2g+1}$, as in Section 2.1. In this case there is no natural section of $\pi : \bar{\mathcal{C}}_g \to \mathcal{P}_{2g+1}$, so we need to choose two sections of π to construct a Bäcklund transformation (for the existence of such sections the remarks from Section 2.3 apply). To simplify the formulas for the Bäcklund transformation and to make them very similar to the formulas in the odd case we pick one of the sections such that every $f \in \mathcal{P}_{2g+1}$ gets mapped to one of the two points at infinity, i.e. in $\bar{\Gamma}_f \setminus \Gamma_f$. We denote this section by ξ_∞ and we pick another section ξ. Since Mumford's prescription (6) and (7) applies unchanged, the following variant to (8) realizes the linear equivalence which is needed in order to express a shift over $[\xi(f) - \xi_\infty(f)]$ on $\mathrm{Jac}(\bar{\Gamma}_f)$:

$$F(x, y) = \frac{y + v(x) + u(x)(\pm(x - \lambda_f) + \beta)}{u(x)(x - \lambda_f)} = \frac{y + v(x) + \beta u(x)}{u(x)(x - \lambda_f)} \pm 1, \tag{44}$$

where β is such that the numerator vanishes at $(\lambda_f, -\mu_f)$, so that

$$\beta = \frac{\mu_f - v(\lambda_f)}{u(\lambda_f)}. \tag{45}$$

The $\pm$ in (44) depends on the chosen section ξ_∞, its actual value, for a given f being determined by expressing x and y in terms of a local parameter at the point $\xi_\infty(f)$. The rest of the computation is similar to the one in Section 2.1, giving

$$\begin{aligned}
\tilde{u}(x) &= \frac{u(x)(x - \lambda_f \pm \beta)^2 \pm 2v(x)(x - \lambda_f \pm \beta) - w(x)}{(u_1 - w_{-1} - 2\lambda_f \pm 2\beta)(x - \lambda_f)},\\
\tilde{v}(x) &= -v(x) \mp u(x)(x - \lambda_f \pm \beta) \pm \tilde{u}(x)(x - \lambda_f + u_1 - \tilde{u}_1 \pm \beta),\\
\tilde{w}(x) &= \frac{u(x)w(x) + v^2(x) - \tilde{v}^2(x)}{\tilde{u}(x)}, \quad \beta = \frac{\mu_f - v(\lambda_f)}{u(\lambda_f)}.
\end{aligned} \tag{46}$$

The value of the variable $\tilde{u}_1$ in terms of the original variables is computed from the first equation in (46) to be given by

$$\tilde{u}_1 = \lambda_f + \frac{u_2 \pm 2v_1 - w_0 \pm 2u_1(\beta \mp \lambda_f) + (\beta \mp \lambda_f)^2}{u_1 - w_{-1} - 2\lambda_f \pm 2\beta}.$$

The matrix $M(x)$, defined as in (13) can in this case be taken as

$$\begin{pmatrix} x - \lambda_f + u_1 - \tilde{u}_1 \pm \beta & \beta(u_1 - \tilde{u}_1 \pm \beta) \pm (x - \lambda_f)(x + \lambda_f + w_{-1} - \tilde{u}_1) \\ \pm 1 & x - \lambda_f \pm \beta \end{pmatrix}. \tag{47}$$

Notice that $\det M(x) = (x - \lambda_f)(u_1 - w_{-1} - 2\lambda_f \pm 2\beta)$.

The integrable vector fields of the even Mumford system are Hamiltonian with respect to a family of Poisson brackets, similar to the brackets (15): if φ is a univariate polynomial of degree at most g then one finds precisely the brackets (15), except for the following two brackets:

$$\{v(x), w(y)\}^{\varphi} = \frac{1}{x-y}(w(x)\varphi(y) - w(y)\varphi(x)) - \alpha(x,y)u(x)\varphi(y),$$
$$\{w(x), w(y)\} = 2\alpha(x,y)(v(x)\varphi(y) - v(y)\varphi(x)), \quad \alpha(x,y) = x + y + w_{-1} - u_1,$$

define a Poisson structure on M_g. Assuming $\varphi(x)$ monic and irreducible, $\varphi(x) = (x - a_1)(x-a_2)\cdots(x-a_g)$, we define an affine map $M_g \to \mathbb{C}^{3g+2}$ by

$$\left(\frac{u(x)}{\varphi(x)}, \frac{v(x)}{\varphi(x)}, \frac{w(x)}{\varphi(x)}\right) = \left(1 + \sum_{i=1}^{g}\frac{f_i}{x-a_i}, \sum_{i=1}^{g}\frac{h_i}{x-a_i}, x^2 + e_{-1}x + e_0 + \sum_{i=1}^{g}\frac{e_i}{x-a_i}\right).$$

As in the case of the Mumford system we find that the variables $\{h_i, e_i, f_i\}_{i=1}^{g}$ are generators for the direct sum of g copies of the Lie–Poisson algebra of $\mathfrak{sl}(2)$. The equation of the spectral curve takes the form

$$\frac{y^2}{\varphi^2(x)} = \frac{f(x)}{\varphi^2(x)} = x^2 + C_{-1}x + C_0 + \sum_{i=1}^{g}\left(\frac{C_i}{(x-a_i)^2} + \frac{H_i}{x-a_i}\right),$$

where $C_i = h_i^2 + e_i f_i$, the Casimir element coming from the ith copy of $\mathfrak{sl}(2)$; moreover $C_{-1} = e_{-1} + \sum_{i=1}^{g} f_i$ and $C_0 = e_0 + \sum_{i=1}^{g} f_i(C_{-1} + a_i) - (\sum_{i=1}^{g} f_i)^2$ are extra Casimirs. Fixing a generic symplectic leaf, these Casimirs are used to eliminate the variables $e_{-1}, \ldots, e_g$ giving the following equations for the map ($i = 1, \ldots, g$):

$$\left(\sum_{i=1}^{g} 2f_i - 2\lambda \pm 2\beta - C_{-1}\right)(\lambda - a_i)f_i\tilde{f}_i + (f_i(a_i - \lambda \pm \beta) \pm h_i)^2 - C_i = 0,$$
$$\tilde{h}_i = -h_i \mp (f_i - \tilde{f}_i)(a_i - \lambda \pm \beta) \pm \tilde{f}_i \sum_{i=1}^{g}(f_j - \tilde{f}_j),$$

and the following equation for β:

$$\beta^2 \pm 2(u_1 - \tilde{u}_1)\beta - \lambda^2 + \lambda(2\tilde{u}_1 - w_{-1} - u_1) - u_1\tilde{u}_1 - w_0 + u_2 + \tilde{u}_1 w_{-1} \pm 2v_1 = 0,$$

where

$$u_1 = \sum_{i=1}^{g}(f_i - a_i), \qquad v_1 = \sum_{i=1}^{g} h_i, \qquad u_2 = \sum_{i<j} a_i a_j - \sum_{i \neq j} a_i f_j,$$
$$w_{-1} = C_{-1} - \sum_{i=1}^{g}(a_i + f_i),$$
$$w_0 = C_0 - C_{-1}\sum_{i=1}^{g}(a_i + f_i) + \left(\sum_{i=1}^{g} f_i\right)^2 + \sum_{i<j} a_i a_j + \sum_{i \neq j} a_i f_j.$$

Using these formulas the verification of (38) and (42) (where $\hat{\mu}_f$ is in this case again defined by $\hat{\mu}_f = \mu_f/\varphi(\lambda_f)$ and it is assumed that λ_f depends on the Casimirs only) is now straightforward (but lengthy). This shows again that our maps B_ξ are Poisson maps and have the spectrality property when λ_f depends on the Casimirs of $\{\cdot,\cdot\}^\varphi$ only.

In order to show that our maps B_ξ give a discretization of the even Mumford system, we proceed as in Section 2.4. We let $\lambda_f = 1/t$ so that the first few terms of β are given by

$$\beta = \mp\frac{1}{t}\left(1+\frac{w_{-1}-u_1}{2}t+\frac{1}{8}(3u_1^2-2u_1w_{-1} - w_{-1}^2-4u_2+4w_0\pm 8v_1)t^2+\mathrm{O}(t^3)\right).$$

A direct substitution in (46) yields

$$\begin{aligned}
\tilde{u}(x) &= u(x) \mp v(x)t + \mathrm{O}(t^2),\\
\tilde{v}(x) &= v(x) \mp \tfrac{1}{2}(-w(x) + u(x)(x^2 + (w_{-1} - u_1)x\\
&\qquad + u_1^2 + w_0 - u_2 - u_1w_{-1}))t + \mathrm{O}(t^2),\\
\tilde{w}(x) &= w(x) \pm v(x)(x^2 + (w_{-1} - u_1)x + u_1^2 + w_0 - u_2 - u_1w_{-1})t + \mathrm{O}(t^2).
\end{aligned}$$

Moreover we can construct the analogs of Mumford's vector fields X_a. We proceed as in Section 2.4, but special care has to be taken because now the curve has two points at infinity, namely ∞_f and the point that corresponds to ∞_f under the hyperelliptic involution; the latter point will be denoted by ∞'_f. Fixing a section ξ, we write $\xi(f) = (a_f, b_f)$ and we do a translation over $[(a_f, -b_f) - \infty_f]$. The matrix going with this transformations is denoted by $P(x)$. Then we translate over $[(\lambda_f(t), \mu_f(t)) - \infty'_f]$; its matrix is denoted by $Q_t(x)$. The product then corresponds to a translation over $[(\lambda_f(t), \mu_f(t)) - (a_f, b_f)]$. Explicitly, for $P(x)$ we take the lower signs in (47) to get

$$P(x) = \begin{pmatrix} x - a + u_1 - \tilde{u}_1 + \beta & \beta(\tilde{u}_1 - u_1 - \beta) - (x - a)(x + a + w_{-1} - \tilde{u}_1) \\ -1 & x - a + \beta \end{pmatrix}$$

with

$$\tilde{u}_1 = a_f + \frac{u_2 - 2v_1 - w_0 - 2u_1(a_f - \beta) + (a_f - \beta)^2}{u_1 - w_{-1} - 2a_f + 2\beta},$$

$$\beta = \frac{b_f + v(a_f)}{u(a_f)} = \frac{w(a_f)}{b_f - v(a_f)}.$$

For $Q_t(x)$ we take the upper sign and we find

$$Q_t(x) = \begin{pmatrix} x - \lambda_f(t) + \tilde{u}_1 - \tilde{u}t_1 + \beta(t) & \star \\ 1 & x - \lambda_f(t) + \beta(t) \end{pmatrix},$$

where $\star = \beta(t)(\tilde{u}_1 - \tilde{u}t_1 + \beta(t)) + (x - \lambda_f(t))(x + \lambda_f(t) + \tilde{w}_{-1} - \tilde{u}t_1)$ and

$$\tilde{\tilde{u}}_1 = \lambda_f(t) + \frac{\tilde{u}_2 + 2\tilde{v}_1 - \tilde{w}_0 + 2\tilde{u}_1(\beta(t) - \lambda_f(t)) + (\beta(t) + \lambda_f(t))^2}{\tilde{u}_1 - \tilde{w}_{-1} - 2\lambda_f(t) + 2\beta(t)},$$
$$\beta(t) = \frac{\mu_f(t) - \tilde{v}(\lambda_f(t))}{\tilde{u}(\lambda_f(t))}.$$

In order to express $\tilde{\tilde{u}}_1$ in terms of the original phase variables, as needed in the computation, one needs explicit formulas for $\tilde{u}_2$, $\tilde{v}_1$, $\tilde{w}_{-1}$ and $\tilde{w}_0$. For $\tilde{u}_2$ and $\tilde{v}_1$ we find by expanding the first Bäcklund transformation in terms powers of t

$$\tilde{u}_2 = a\tilde{u}_1 + \frac{u_3 - 2(a-\beta)u_2 + (a-\beta)^2 u_1 - 2v_2 + 2(a-\beta)v_1 - w_1}{u_1 - w_{-1} - 2a_f + 2\beta},$$
$$\tilde{v}_1 = -v_1 + u_2 - (a-\beta)u_1 - \tilde{u}_2 + \tilde{u}_1(a - u_1 + \tilde{u}_1 - \beta).$$

We find as in the case of the Mumford system that $\beta(0) = u_1 - \tilde{u}_1 + \beta$ and that

$$\beta'(0) = 1 - (u_1 - w_{-1} - 2a + 2\beta)\frac{u(a)}{2b}.$$

As we have seen in the Mumford case the vector field which corresponds to the deformation family is given by

$$\frac{\mathrm{d}L}{\mathrm{d}t_{a_f}}(x) = [Q_0'(x)\, Q_0^{-1}(x), L(x)],$$

which leads by direct substitution to

$$\frac{\mathrm{d}L}{\mathrm{d}t_{a_f}}(x) = \frac{1}{2b_f}\left[\frac{L(a_f)}{x - a_f} + \begin{pmatrix} 0 & u(a_f)(x + a_f + u_1 - w_{-1}) \\ 0 & 0 \end{pmatrix}, L(x)\right].$$

As far as we could check these vector fields are new.

3.2. Generalized Jacobians (odd case)

We now consider a first case in which the fibers of the moment map are affine parts of generalized (hyperelliptic) Jacobians. The main difference between the generalized Jacobian case and the usual case is that generalized Jacobians have a larger symmetry group, leading to more general Bäcklund transformations.

We first define phase space, which is denoted by $\hat{M}_g$, a moment map $\hat{\chi} : \hat{M}_g \to \mathcal{P}_{2g+1}$, we construct a natural map $\pi : \hat{M}_g \to M_g$ onto the phase space of the Mumford system, and we give a geometric description of the fibers of χ. For any $g \geq 1$, $\hat{M}_g$ is the space of all Lax matrices of the form

$$L(x) = \begin{pmatrix} V(x) & W(x) \\ U(x) & -V(x) \end{pmatrix},$$

where the entries of $L(x)$ are now subject to the following constraints: $U(x)$ and $W(x)$ are monic and their degrees are respectively g and $g+1$; the degree of $V(x)$ is at most g. Writing

$$\begin{aligned} U(x) &= x^g + U_1 x^{g-1} + \cdots + U_g, \qquad V(x) = V_0 x^g + \cdots + V_g, \\ W(x) &= x^{g+1} + W_0 x^g + \cdots + W_g, \end{aligned}$$

we take the coefficients of these three polynomials as coordinates on $\hat{M}_g$. It is clear that the group of matrices of the form

$$N_\tau = \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} \tag{48}$$

acts on $\hat{M}_g$ by the adjoint action, where τ is any function on $\hat{M}_g$. In particular, taking $\tau = V_0$ we get a map onto a subspace which is exactly the phase space M_g of the Mumford system; we denote this natural map by π and denote the composition $\chi \circ \pi$ by $\hat{\chi}$; explicitly $\hat{\chi}$ is given by $L(x) \mapsto -\det L(x)$. For $f \in \mathcal{P}_{2g+1}$ such that Γ_f is smooth the fiber $\chi^{-1}(f)$ is an affine part of $\mathrm{Sym}^{g+1} \bar{\Gamma}_f$, the $(g+1)$th symmetric product of $\bar{\Gamma}_f$ (recall that $\bar{\Gamma}_f$ has genus g). To see this, one associates to $(U(x), V(x), W(x)) \in \chi^{-1}(f)$ the divisor $D = \sum_{i=1}^{g+1} (x_i, y_i)$, where x_i are the roots of $W(x)$ and $y_i = -V(x_i)$. It is easy to show that this realizes a bijection between $\chi^{-1}(f)$ and an affine part of $\mathrm{Sym}^{g+1}(\bar{\Gamma}_f)$.[7] The rational function

$$\frac{y - V(x)}{W(x)} = \frac{U(x)}{y + V(x)}$$

shows that $\mathcal{D}$ is linearly equivalent to the divisor $\mathcal{D}' + \infty_f = \sum_{i=1}^{g} (x_i', y_i') + \infty_f$, where x_i' are the zeros of $U(x)$ and $V(x_i') = y_i'$ for $i = 1, \ldots, g$. This gives a geometric interpretation of the map π, and it shows that, under the above correspondence between points of $\hat{M}_g$ and divisors, the adjoint action by N_τ maps divisors to linearly equivalent divisors.

We will show that this geometric picture leads, via our geometric construction of Bäcklund transformations, to a family of Bäcklund transformations $B_{\xi,\alpha} : \hat{M}_g \to \hat{M}_g$ which makes the following diagram commutative:

$$\begin{tikzcd} \hat{M}_g \arrow[r, dotted, "B_{\xi,\alpha}"] \arrow[d, "\pi"'] & \hat{M}_g \arrow[d, "\pi"'] \\ M_g \arrow[r, "B_\xi"] & M_g \end{tikzcd} \tag{49}$$

It should be clear that, since we are forced to work with divisors, we cannot write (5) as a *definition* for $B_{\xi,\alpha}$ because the effective divisor of degree $g+1$ that corresponds to a line bundle of degree $g+1$ is not unique. Accordingly we write down a general formula for a map satisfying (5) and then we specialize the arbitrary function that figures in it so as to

[7] From this description it follows easily that the fiber $\chi^{-1}(f)$ can also be described as an affine part of the generalized Jacobian of Γ_f with respect to the divisor $2\infty_f$ (see [20]).

obtain a Bäcklund transformation. Explicitly, we let $\xi(f) = (\lambda_f, \mu_f, f)$, as before, and we consider for a generic point $(U(x), V(x), W(x)) \in \hat{M}_g$ the following function:

$$F(x, y) = \frac{(y - V(x))(x - \lambda_f + \alpha\beta) + \alpha W(x)}{W(x)(x - \lambda_f)}.$$

We have chosen a combination of the parameters α and β such that, when we express that the numerator of F vanishes at $(\lambda_f, -\mu_f)$, then we find

$$\beta = \frac{W(\lambda_f)}{\mu_f + V(\lambda_f)} = \frac{\mu_f - V(\lambda_f)}{U(\lambda_f)},$$

so that β is formally given by the same formula (9) as in the Mumford system. With this choice of β we find for any α that $F(x, y)$ has $\mathcal{D}+(\lambda_f, \mu_f) = \sum_{i=1}^{g+1}(x_i, y_i)+(\lambda_f, \mu_f)$ as its polar divisor and vanishes at infinity. It follows that the other zeros of $F(x, y)$ give a divisor $\tilde{\mathcal{D}}$ which is linearly equivalent to the divisor $\mathcal{D}$ which is associated to $(U(x), V(x), W(x))$, up to a shift over $(\lambda_f, \mu_f)-\infty_f$. Multiplying $F(x, y)$ by $(y+V(x))(x-\lambda_f+\alpha\beta)-\alpha W(x)$ and using $y^2 = U(x)W(x) + V^2(x)$ we find an equation for the x-coordinates of the image divisor and we deduce, as in the case of the Mumford system,

$$\tilde{W}(x) = -\frac{(x - \lambda_f + \alpha\beta)^2 U(x) + 2\alpha(x - \lambda_f + \alpha\beta)V(x) - \alpha^2 W(x)}{\lambda_f - x}. \tag{50}$$

By interpolation at the zeros of $\tilde{W}$ we also find

$$\tilde{V}(x) = \frac{\beta(x - \lambda_f + \alpha\beta)U(x) + (x - \lambda_f + 2\alpha\beta)V(x) - \alpha W(x)}{\lambda_f - x}, \tag{51}$$

and the formula for $\tilde{U}(x)$ follows from $\tilde{U}(x)\tilde{W}(x) + \tilde{V}^2(x) = U(x)W(x) + V^2(x)$,

$$\tilde{U}(x) = \frac{\beta^2 U(x) + 2\beta V(x) - W(x)}{\lambda_f - x}. \tag{52}$$

This gives explicit formulas for the map $B_{\xi,\alpha}$. In terms of matrices, $B_{\xi,\alpha}$ is given by $L \mapsto MLM^{-1}$, where M can be taken as follows:

$$M(x) = \begin{pmatrix} \alpha & x - \lambda_f + \alpha\beta \\ 1 & \beta \end{pmatrix}. \tag{53}$$

The commutativity of (49) is a direct consequence of the equality $N_{-V_0+\alpha-\beta}M = \bar{M}N_{V_0}$, where $\bar{M}$ is given by

$$\bar{M}(x) = \begin{pmatrix} \beta + V_0 & x - \lambda_f + (\beta + V_0)^2 \\ 1 & \beta + V_0 \end{pmatrix}.$$

If we compare (14) and (53) then we see that both matrices coincide when $\alpha = \beta$, but, as we will see, the choice $\alpha = \beta$ does not lead to a Bäcklund transformation (when $\alpha = \beta$ it is not a Poisson map).

We now come to Poissonicity of the maps that we have constructed. The Poisson structure of the generalized Mumford system is given, in the notation of Section 2.2, by

$$\{L(x) \overset{\otimes}{,} L(y)\} = [r(x-y), L_1(x)\varphi(y) + \varphi(x)L_2(y)], \tag{54}$$

where $\varphi(x)$ is a polynomial of at most degree g. We take λ_f to be dependent on the Casimirs only and we compute, as before, the brackets with β, giving

$$\begin{aligned}
\{U(x), \beta\}^{\varphi} &= \frac{\mu_f \varphi(x) - \varphi(\lambda_f)(V(x) + \beta U(x))}{\mu_f (x - \lambda_f)},\\
\{V(x), \beta\}^{\varphi} &= -\frac{2\mu_f \beta \varphi(x) - \varphi(\lambda_f)(\beta^2 U(x) + W(x))}{2\mu_f (x - \lambda_f)},\\
\{W(x), \beta\}^{\varphi} &= -\beta \frac{\beta \mu_f \varphi(x) + \varphi(\lambda_f)(\beta V(x) - W(x))}{\mu_f (x - \lambda_f)}.
\end{aligned} \tag{55}$$

Using these formulas we can determine for which choices of α (which could, a priori, be any function on phase space) the map $(U(x), V(x), W(x)) \to (\tilde U(x), \tilde V(x), \tilde W(x))$ is a Poisson map. A (quite long) computation leads to the following conditions on α:

$$\begin{aligned}
&\{\alpha, U(x)\} = -C\frac{V(x) + \beta U(x)}{x - \lambda_f}, \qquad \{\alpha, V(x)\} = -C\frac{W(x) + \beta^2 U(x)}{2(x - \lambda_f)} + D,\\
&\{\alpha, W(x)\} = C\beta\frac{W(x) - \beta V(x)}{x - \lambda_f} + \varphi(x), \qquad \{\alpha, \beta\} = \frac{\varphi(\lambda_f)}{(2\mu_f)}.
\end{aligned}$$

In these formulas C and D are any functions on phase space. However, since the left-hand side of the first three expressions is polynomial in x, the same must be true for the right-hand side, which implies that $C = 0$. Using the last equation and the definition of β we find that $D = 0$ and we are left with

$$\{\alpha, U(x)\} = \{\alpha, V(x)\} = 0, \qquad \{\alpha, W(x)\} = \varphi(x), \qquad \{\alpha, \beta\} = \frac{\varphi(\lambda_f)}{2\mu_f}. \tag{56}$$

It turns out that there is such an α, namely $\alpha = V_0$; to obtain the most general solution it suffices to add any Casimir of φ to V_0. A direct check that one gets for those values of α indeed a Poisson map can be done quite easily by using the following formulas, which follow from (54)–(56):

$$\begin{aligned}
\{L(x) \overset{\otimes}{,} M(y)\} &= \left(\frac{\varphi(\lambda_f)}{2\mu_f}[L(x), N(x)] + \varphi(x)N(x)\right) \otimes \frac{\partial M}{\partial \beta} - \varphi(x)\frac{\partial^2 M}{\partial\alpha\partial\beta} \otimes \frac{\partial M}{\partial \alpha},\\
\{M(x) \overset{\otimes}{,} L(y)\} &= -\frac{\partial M}{\partial \beta} \otimes \left(\frac{\varphi(\lambda_f)}{2\mu_f}[L(y), N(y)] + \varphi(y)N(y)\right) + \varphi(y)\frac{\partial M}{\partial \alpha} \otimes \frac{\partial^2 M}{\partial\alpha\partial\beta},\\
\{M(x) \overset{\otimes}{,} M(y)\} &= -\frac{\varphi(\lambda_f)}{2\mu_f}\begin{pmatrix} 0 & \alpha & -\alpha & 0\\ 0 & 1 & 0 & \beta\\ 0 & 0 & -1 & -\beta\\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

where

$$N(x)=1/\lambda_f - x\begin{pmatrix}\beta & \beta^2\\ -1 & -\beta\end{pmatrix}.$$

In conclusion we have shown that when λ_f and $\alpha - V_0$ depend only on the Casimirs then the map $B_{\xi,\alpha}$ is a Bäcklund transformation for the generalized Mumford system.

In order to check spectrality of the map $B_{\xi,\alpha}$ when λ_f and $\alpha - V_0$ depend only on the Casimirs one proceeds as in the case of the Mumford system. We fix a monic polynomial $\varphi(x)$ of degree g with distinct roots $a_1,\dots,a_g$ and we define an affine map $\hat{M}_g \to \mathbb{C}^{3g+2}$ by

$$\left(\frac{U(x)}{\varphi(x)},\frac{V(x)}{\varphi(x)},\frac{W(x)}{\varphi(x)}\right) = \left(1+\sum_{i=1}^{g}\frac{f_i}{x-a_i}, h_0+\sum_{i=1}^{g}\frac{h_i}{x-a_i}, x+f_0+\sum_{i=1}^{g}\frac{e_i}{x-a_i}\right). \tag{57}$$

In this case we get the brackets (33) with addition one non-trivial bracket, $\{h_0, f_0\}=1$. We denote the Casimir element coming from the ith copy of $\mathfrak{sl}(2)$ by C_i, $C_i = h_i^2 + e_i f_i$ and we denote the Casimir $\alpha - V_0$ by C. We fix a symplectic leaf and we express the variables $f_0,\dots,f_g,\tilde{f}_0,\dots,\tilde{f}_g$ in terms of $h_0,\dots,h_g,\tilde{h}_0,\dots,\tilde{h}_g$ and λ. To do this, first notice that

$$\alpha = h_0 + C \quad\text{and}\quad \beta = C - \tilde{h}_0,$$

as follows easily from (51) and (57). The formulas for the variables $f_1,\dots,f_g$ follow from

$$(\beta f_i + h_i)^2 - f_i\tilde{f}_i(\lambda - a_i) - C_i = 0, \tag{58}$$

$$\tilde{h}_i + h_i - \alpha\tilde{f}_i + \beta f_i = 0, \tag{59}$$

which one derives from Eqs. (50)–(52) for B_ξ, expressed in terms of the variables f_i and h_i. Indeed, if we use the second equation to eliminate $\tilde{f}_i$ from the first equation we get

$$f_i^2\tilde{h}_0(a_i - \lambda - h_0\tilde{h}_0) + f_i((\lambda - a_i)(\tilde{h}_i + h_i) + 2h_i h_0\tilde{h}_0) + h_0(C_i - h_i^2) = 0, \tag{60}$$

and this defines $f_1,\dots,f_g$ as a function of the variables h_j and $\tilde{h}_j$; the second equation in (58) then defines $\tilde{f}_1,\dots,\tilde{f}_g$ as a function of these variables. As for f_0 and $\tilde{f}_0$, they are given by

$$f_0 = -\lambda + \sum_{i=1}^{g}\tilde{f}_i + \tilde{h}_0^2 - 2h_0\tilde{h}_0, \qquad \tilde{f}_0 = -\lambda + \sum_{i=1}^{g} f_i + h_0^2 - 2h_0\tilde{h}_0,$$

as follows also from (50)–(52). Using these formulas it is straightforward to verify the following integrability conditions ($i, j = 1, \dots, g$):

$$\begin{aligned}
\frac{\partial f_i}{\partial \lambda} &= -f_i \frac{\partial \hat{\mu}}{\partial h_i} = -\frac{\alpha f_i \tilde{f}_i}{(\lambda - a_i)(\beta f_i + \alpha \tilde{f}_i) - 2\alpha\beta(h_i + \beta f_i)},\\
\frac{\partial \tilde{f}_i}{\partial \lambda} &= \tilde{f}_i \frac{\partial \hat{\mu}}{\partial h_i} = -\frac{\beta f_i \tilde{f}_i}{(\lambda - a_i)(\beta f_i + \alpha \tilde{f}_i) - 2\alpha\beta(h_i + \beta f_i)},\\
\frac{\partial f_0}{\partial \lambda} &= -\frac{\partial \hat{\mu}}{\partial h_0} = -1 - \beta \sum_{i=1}^{g} \frac{f_i \tilde{f}_i}{(\lambda - a_i)(\beta f_i + \alpha \tilde{f}_i) - 2\alpha\beta(h_i + \beta f_i)},\\
\frac{\partial \tilde{f}_0}{\partial \lambda} &= \frac{\partial \hat{\mu}}{\partial \tilde{h}_0} = -1 - \alpha \sum_{i=1}^{g} \frac{f_i \tilde{f}_i}{(\lambda - a_i)(\beta f_i + \alpha \tilde{f}_i) - 2\alpha\beta(h_i + \beta f_i)}.
\end{aligned}$$

This shows that the maps B_ξ have the spectrality property. In the same way one can verify the compatibility conditions

$$f_j \frac{\partial f_i}{\partial h_j} = f_i \frac{\partial f_j}{\partial h_i}, \qquad \tilde{f}_j \frac{\partial \tilde{f}_i}{\partial \tilde{h}_j} = \tilde{f}_i \frac{\partial \tilde{f}_j}{\partial \tilde{h}_i}, \qquad f_j \frac{\partial \tilde{f}_i}{\partial h_j} = -\tilde{f}_i \frac{\partial f_j}{\partial \tilde{h}_i},$$

giving an alternative proof that the maps B_ξ are Poisson maps.

We now show that these Bäcklund transformation discretize the underlying integrable system. The computation is similar as in the previous cases, except that one has to choose the Casimir $\alpha - V_0$ carefully so as to obtain the identity transformation in the limit $\lambda_f \to \infty$. Since the point at infinity of the curve is a Weierstraß point we let $\lambda = t^{-2}$ and we choose $\alpha = V_0 + 1/t$. Then

$$\beta = \frac{1}{t} - V_0 + \frac{1}{2}(W_0 - U_1 + V_0^2)t + \mathrm{O}(t^2),$$

and we find by direct substitution

$$\begin{aligned}
\tilde{U}(x) &= U(x) + 2t(V(x) - V_0 U(x)) + \mathrm{O}(t^2),\\
\tilde{V}(x) &= V(x) + t(U(x)\tfrac{1}{2}(2x + W_0 - U_1 - V_0^2) - W(x)) + \mathrm{O}(t^2),\\
\tilde{W}(x) &= W(x) - t(V(x)(2x + W_0 - U_1 - V_0^2) - 2V_0 W(x)) + \mathrm{O}(t^2),
\end{aligned}$$

from which we can read off the vector field. For the vector fields X_a the computation is very similar to the one in the case of the Mumford system. Namely we take

$$P(x) = \begin{pmatrix} \alpha & x - a_f - \alpha\beta \\ 1 & -\beta \end{pmatrix}$$

with $\alpha = V_0$ and $\beta = (b_f + v(a_f))/u(a_f)$; moreover we take

$$Q_t(x) = \begin{pmatrix} \alpha(t) & x - \lambda_f(t) + \alpha\beta(t) \\ 1 & \beta(t) \end{pmatrix},$$

where $\alpha(t) = \tilde{V}_0 = \beta$ (so that in fact α is independent of t) and $\beta(t) = (\mu_f(t) - \tilde{V}(\lambda_f(t)))/\tilde{U}(\lambda_f(t))$, so that $\beta(0) = -\alpha$. Using $\beta'(0) = U(a)/(2b_f)$ we find

$$Q_0'(x)Q_0^{-1}(x) = \frac{1}{2b_f(x-a)}\begin{pmatrix} V(a) - b_f & W(a) \\ U(a) & -V(a) \end{pmatrix},$$

so that, after removal of a diagonal matrix, we find the following Lax equation:

$$\frac{\mathrm{d}L}{\mathrm{d}t_{a_f}}(x) = \frac{1}{2b_f}\left[\frac{L(a_f)}{x-a_f}, L(x)\right].$$

We shortly indicate how the above maps $B_{\xi,\alpha}$ can also be found from the eigenvectors of the Lax operator. Taking $\vec{\alpha}_0 = (1, 0)$ and $\vec{\alpha} = (\gamma, \delta - x)$ we express that the solutions to

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -\tilde{v}(x) - y & -\tilde{w}(x) \\ -\tilde{u}(x) & \tilde{v}(x) - y \end{pmatrix} = 0, \tag{61}$$

are the same as the solutions of

$$\begin{pmatrix} \gamma & \delta - x \end{pmatrix}\begin{pmatrix} -v(x) - y & -w(x) \\ -u(x) & v(x) - y \end{pmatrix} = 0, \tag{62}$$

except that (62) also has $(\lambda_f, -\mu_f)$ as a solution. By eliminating y from (62) we find that

$$\tilde{W}(x) = \frac{(x-\delta)^2 U(x) - 2(x-\delta)\gamma V(x) - \gamma^2 W(x)}{x - \lambda_f},$$

because the numerator of the above right-hand side is monic of degree $g + 2$ and vanishes at the roots of W as well as at $x = \lambda_f$. By interpolation at the zeros of $\tilde{W}$ we find that

$$\tilde{V}(x) = \frac{(x-\delta)(\delta-\lambda_f)U(x) + (2\delta - \lambda_f - x)\gamma V(x) + \gamma^2 W(x)}{\gamma(x-\lambda_f)}.$$

We recover our formulas (50) and (51) (hence also (52)) by taking $\gamma = \alpha$ and $\delta = \lambda - \alpha\beta$.

3.3. *Generalized Jacobians (even case)*

In this case phase space $\hat{M}_g$ is given by the space of triples of polynomials $(U(x), V(x), W(x))$ with the following degree constraints:

$$U(x) = x^{g+1} + U_0 x^g + \cdots + U_g, \qquad V(x) = V_0 x^g + \cdots + V_g,$$
$$W(x) = x^{g+1} + W_0 x^g + \cdots + W_g.$$

In this case the spectral curve is of the form $y^2 = f(x)$, where $f(x) = U(x)W(x) + V^2(x)$ is monic of degree $2g + 2$. When f is irreducible the corresponding fiber of the moment map $\hat{\chi}$ (which is given as in the other cases by $\hat{\chi}(L(x)) = -\det L(x)$) is an affine part of $\mathrm{Sym}^{g+1}\bar{\Gamma}_f$; this is shown by associating to $(U(x), V(x), W(x)) \in \chi^{-1}(f)$ the divisor $D = \sum_{i=1}^{g+1}(x_i, y_i)$, where x_i are the roots of $U(x)$ and $y_i = V(x_i)$. We choose a section ξ and we let $\xi(f) = (\lambda_f, \mu_f, f)$. For a generic point $(U(x), V(x), W(x)) \in \hat{M}_g$ we consider the function

$$F(x, y) = \frac{(y + V(x))(x - \alpha_1) \pm U(x)(x - \alpha_2)}{U(x)(x - \lambda_f)},$$

where α_1 and α_2 satisfy the following linear equation (zero of the numerator of F at the point $(\lambda_f, -\mu_f)$):

$$(-\mu_f + V(\lambda_f))(\lambda_f - \alpha_1) \pm U(\lambda_f)(\lambda_f - \alpha_2) = 0.$$

$F(x, y)$ has $\mathcal{D} + (\lambda_f, \mu_f) = \sum_{i=1}^{g+1}(x_i, y_i) + (\lambda_f, \mu_f)$ as its polar divisor and vanishes at infinity. It follows that the other zeros of $F(x, y)$ give a divisor $\tilde{\mathcal{D}}$ which is linearly equivalent to the divisor $\mathcal{D}$ which is associated to $(U(x), V(x), W(x))$, up to a shift over $(\lambda_f, \mu_f) - \infty_f$. It leads to the following formulas for the map B_ξ:

$$\begin{aligned}
\tilde{U}(x) &= \frac{U(x)(x-\alpha_2)^2 \pm 2V(x)(x-\alpha_1)(x-\alpha_2) - W(x)(x-\alpha_1)^2}{C(x-\lambda_f)},\\
\tilde{V}(x) &= \frac{1}{C(x-\lambda_f)}[\pm(x-\alpha_2)(x-\alpha_4)U(x) + ((x-\alpha_2)(x-\alpha_3)\\
&\quad + (x-\alpha_1)(x-\alpha_4))V(x) \mp (x-\alpha_1)(x-\alpha_3)W(x)],\\
\tilde{W}(x) &= \frac{-U(x)(x-\alpha_4)^2 \mp 2V(x)(x-\alpha_3)(x-\alpha_4) + W(x)(x-\alpha_3)^2}{C(x-\lambda_f)},
\end{aligned}$$

where

$$C = 2(\alpha_1 - \alpha_2) + U_0 \pm 2V_0 - W_0, \tag{63}$$

and

$$\alpha_3 = \alpha_1 - C\frac{\alpha_1 - \lambda_f}{\alpha_1 - \alpha_2}, \qquad \alpha_4 = \alpha_2 - C\,\frac{\alpha_2 - \lambda_f}{\alpha_1 - \alpha_2}. \tag{64}$$

The above transformation can be rewritten in the form of the matrix equation $M(x)L(x) = \tilde{L}(x)M(x)$ with the following matrix M:

$$M(x) = \begin{pmatrix} x - \alpha_3 & \pm(x - \alpha_4) \\ \pm(x - \alpha_1) & x - \alpha_2 \end{pmatrix}, \tag{65}$$

where the variables $\alpha_1, \ldots, \alpha_4$ are given by

$$\alpha_i = \lambda_f + \frac{(\epsilon_i C - U_0 \mp 2V_0 + W_0)((-1)^{i-1}C - \tilde{U}_0 \pm 2\tilde{V}_0 + \tilde{W}_0)}{4C}, \tag{66}$$

where $\epsilon_i = 1$ for $i = 1, 2$ and $\epsilon_i = -1$ otherwise.

Let us now turn to Poissonicity and spectrality. For every polynomial φ of degree at most $g + 1$ we find a Poisson structure $\{\cdot, \cdot\}^\varphi$ which is given formally by precisely the same formulas as in the case considered in Section 3.2. We can see from the above formulas that it will be much easier to do further calculations if we make a simple similarity transform:

$$M(x) \mapsto SM(x)S^{-1}, \qquad L(x) \mapsto SL(x)S^{-1}, \tag{67}$$

where

$$S = \begin{pmatrix} 1 & \pm 1 \\ 1 & \mp 1 \end{pmatrix}. \tag{68}$$

Let us denote the transformed matrices $L(x)$ and $M(x)$ by small letters $\ell(x)$ and $m(x)$, respectively,

$$\ell(x) = SL(x)S^{-1}, \qquad \tilde{\ell}(x) = S\tilde{L}(x)S^{-1}, \qquad m(x) = SM(x)S^{-1},$$

and correspondingly,

$$\ell(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix}, \qquad \tilde{\ell}(x) = \begin{pmatrix} \tilde{v}(x) & \tilde{w}(x) \\ \tilde{u}(x) & -\tilde{v}(x) \end{pmatrix}.$$

The triple of new polynomials is as follows:

$$u(x) = u_0 x^g + \cdots + u_g, \qquad v(x) = \pm x^{g+1} + v_0 x^g + \cdots + v_g,$$
$$w(x) = w_0 x^g + \cdots + w_g,$$

and the matrix $m(x)$ has the following form:

$$m(x) = \frac{1}{2C}\begin{pmatrix} 4(C(x-\lambda) + w_0\tilde{u}_0) & \pm 2Cw_0 \\ \pm 2C\tilde{u}_0 & C^2 \end{pmatrix}.$$

Note that the determinant of the matrix $m(x)$, as well as of the matrix $M(x)$, is expressed in terms of C:

$$\det M(x) = \det m(x) = C(x-\lambda).$$

Suppose now that the polynomial $\varphi(x)$ is monic and has distinct roots $a_0, \ldots, a_g$ and consider the map defined by

$$\frac{1}{\varphi(x)}(u(x), v(x), w(x)) = \left(\sum_{i=0}^{g} \frac{f_i}{x-a_i}, \pm 1 + \sum_{i=0}^{g} \frac{h_i}{x-a_i}, \sum_{i=0}^{g} \frac{e_i}{x-a_i}\right).$$

It is an isomorphism between $\hat{M}_g$, equipped with the Poisson structure $\{\cdot,\cdot\}^{\varphi}$, and the direct sum of $g+1$ copies of the Lie–Poisson algebra of $\mathfrak{sl}(2)$. Notice that $u_0 = \sum_{i=0}^{g} f_i$ and $w_0 = \sum_{i=0}^{g} e_i$, so that $m(x)$ depends only on variables e_i and $\tilde{f}_i$. Therefore we take $(e_i, \tilde{f}_i)$, $i = 0, \ldots, g$, as independent variables. Then, it is easy to find the following formulas for the variables $(h_j, \tilde{h}_j)$, $j = 0, \dots, g$:

$$
\begin{aligned}
&C^2 h_j^2 \mp 4C\tilde{u}_0 e_j h_j + 4e_j(C(a_j - \lambda)\tilde{f}_j + \tilde{u}_0^2 e_j) - C^2 C_j = 0,\\
&C^2 \tilde{h}_j^2 \mp 4C w_0 \tilde{f}_j \tilde{h}_j + 4\tilde{f}_j(C(a_j - \lambda)e_j + w_0^2 \tilde{f}_j) - C^2 C_j = 0,\\
&C(h_j - \tilde{h}_j) = \pm 2(\tilde{u}_0 e_j - w_0 \tilde{f}_j).
\end{aligned}
\tag{69}
$$

As for the compatibility conditions:

$$
e_k \frac{\partial h_j}{\partial e_k} = e_j \frac{\partial h_k}{\partial e_j}, \qquad \tilde{f}_k \frac{\partial \tilde{h}_j}{\partial \tilde{f}_k} = \tilde{f}_j \frac{\partial \tilde{h}_k}{\partial \tilde{f}_j}, \qquad \tilde{f}_k \frac{\partial h_j}{\partial \tilde{f}_k} = e_j \frac{\partial \tilde{h}_k}{\partial e_j},
$$

we have from (69) that

$$
\frac{\partial h_j}{\partial e_k} = \frac{\partial \tilde{h}_j}{\partial \tilde{f}_k} = 0, \quad j \neq k,
$$

which leads at once to the first two equations and to the third equation for $j \neq k$. The proof of the third equation for $i = k$ is easy by direct computation. The spectrality property also holds, as one easily verifies the following formulas:

$$
e_j \frac{\partial \hat{\mu}}{\partial e_j} = -\frac{\partial h_j}{\partial \lambda}, \qquad \tilde{f}_j \frac{\partial \hat{\mu}}{\partial \tilde{f}_j} = -\frac{\partial \tilde{h}_j}{\partial \lambda},
$$

where $\hat{\mu} = \mu/\varphi(\lambda)$.

We finish by computing the continuum flows, obtained by taking the limit $t \to 0$ of the family of sections ξ_t given by $\lambda = 1/t$ and $\mu = \mp(1 + (U_0 + W_0)t/2 + \mathrm{O}(t^2))/t^{g+1}$. In order for the limit to exist we must take the Casimir C of the form $C' - 4\lambda$, where C' does not depend on λ. Then

$$
a_1 = \frac{C' - 2U_0 + 2(\pm 1 - 1)V_0 + 2W_0}{4}, \qquad a_2 = \frac{2}{t} - \frac{C' - 2(1 \pm 1)V_0}{4},
$$

and in the limit our Bäcklund transformations lead, as in the other cases, to a vector field which has the Lax form $L'(x) = [L(x), N(x)]$, where $N(x)$ is given (up to a constant factor 1/8) by

$$
\begin{pmatrix}
(2 \pm 2)V_0 & \pm(4x - C' - 2U_0 - (2 \mp 2)V_0 + 2W_0)\\
\pm(4x - C' + 2U_0 + (2 \mp 2)V_0 - 2W_0) & -(2 \pm 2)V_0
\end{pmatrix}.
$$

In terms of $l(x)$ this becomes $l'(x) = [l(x), n(x)]$, where $n(x) = VN(x)V^{-1}$ is given by

$$
n(x) = \pm\frac{1}{8}\begin{pmatrix} 4x - C' & 4w_0 \\ 4u_0 & C' - 4x \end{pmatrix}.
$$

The above vector fields is the analog of the vector field X_∞ of the Mumford system. The analogs of the vector fields $X_a, a \in \mathbb{P}^1$ are constructed in the same way as in the other cases.

3.4. Geodesic flow on SO(4)

We now look at the case of an integrable geodesic flow on SO(4), whose underlying metric appears as metric II in the classification of integrable geodesic flows on SO(4). In suitable coordinates, the basic vector field X_1 of this a.c.i. system is given by the differential equations

$$\dot{z}_1 = 2z_5z_6, \qquad \dot{z}_2 = 2z_3z_4, \qquad \dot{z}_3 = z_5(z_1+z_4),$$
$$\dot{z}_4 = 2z_2z_3, \qquad \dot{z}_5 = z_3(z_1+z_4), \qquad \dot{z}_6 = 2z_1z_5,$$

and it admits the following quadratic first integrals:

$$H_1 = z_3^2 - z_5^2, \qquad H_2 = z_1^2 - z_6^2, \qquad H_3 = z_2^2 - z_4^2,$$
$$H_4 = (z_1+z_4)^2 + 4(z_3^2 - z_2z_5 - z_3z_6). \tag{70}$$

Following [5] we let

$$u(x) = x^2 + \left(\frac{z_1+z_2+z_4+z_6}{2(z_3-z_5)} - 1\right)x - \frac{z_2+z_4}{2(z_3-z_5)},$$

and we let $v(x)$ be the polynomial of degree at most 1, characterized by

$$v(0) = u(0)(z_1+z_4+2z_3), \qquad v(1) = u(1)(z_1+z_4+2z_5).$$

This map associates to any point P in $\mathbb{C}^6$ an unordered pair of points on the algebraic curve

$$\Gamma : y^2 = f(x) = x(1-x)[4x^3h_1 - (4h_1+h_4)x^2 + (h_4-h_3-h_2)x + h_3], \tag{71}$$

where h_i denotes the value of H_i at P. Notice that the polynomial f which defines Γ is not monic, its leading term being dependent on the integrals. As a consequence, the polynomial w, defined by $w(x) = f(x) - v^2(x)/u(x)$, will not be monic and the map does *not* define a map to the Mumford system (indeed, for most of the Poisson structures of this system this leading term is not even a Casimir). For future use, notice that $w(0) = -u(0)(z_1+z_4+2z_3)^2$ and $w(1) = -u(1)(z_1+z_4+2z_5)^2$, because f has 0 and 1 as roots. Conversely, given three such polynomials u, v, w which satisfy $v^2(x) + u(x)w(x) = f(x)$, where f has the above form (71), the corresponding point $(z_1, \ldots, z_6) \in \mathbb{C}^6$ is reconstructed by using the following formulas:

$$z_3 - z_5 = \frac{1}{2}\left(\frac{v(0)}{u(0)} - \frac{v(1)}{u(1)}\right), \qquad z_2 + z_4 = \left(\frac{v(1)}{u(1)} - \frac{v(0)}{u(0)}\right)u(0),$$
$$z_1 + z_6 = \left(\frac{v(0)}{u(0)} - \frac{v(1)}{u(1)}\right)u(1), \tag{72}$$

in addition to the first three equations in (70).

In order to construct Bäcklund transformations for this system we consider, for a fixed point $P \in \mathbb{C}^6$, the following rational function:

$$F(x, y) = \frac{y + v(x) + \beta u(x)}{u(x)(x - \lambda_f)}, \tag{73}$$

and we demand that the numerator of F vanishes at the point $(\lambda_f, -\mu_f)$, as in the case of the Mumford system. It leads to

$$\tilde{u}(x) = \frac{\beta^2 u(x) + 2\beta v(x) - w(x)}{-4h_1(\lambda_f - x)},$$
$$\tilde{v}(x) = \frac{(\beta^3 - 4h_1\beta(x-\lambda_f))u(x) + (2\beta^2 - 4h_1(x-\lambda_f))v(x) - \beta w(x)}{4h_1(x-\lambda_f)}, \tag{74}$$

the value of $\tilde{w}(x)$ is not needed for the computation. Writing (72) in terms of tilded variables and substituting (74) in it we find

$$\frac{\tilde{z}_3 - \tilde{z}_5}{z_3 - z_5} = 2(z_3 + z_5)\left(\frac{\lambda_f}{z_1 + z_4 + 2z_3 + \beta} - \frac{\lambda_f - 1}{z_1 + z_4 + 2z_5 + \beta}\right),$$
$$\frac{\tilde{z}_2 + \tilde{z}_4}{z_2 + z_4} = -\frac{1}{4}\frac{\tilde{z}_3 - \tilde{z}_5}{z_3 - z_5}\frac{(z_1 + z_4 + 2z_3 + \beta)^2}{h_1\lambda_f},$$
$$\frac{\tilde{z}_1 + \tilde{z}_6}{z_1 + z_6} = -\frac{1}{4}\frac{\tilde{z}_3 - \tilde{z}_5}{z_3 - z_5}\frac{(z_1 + z_4 + 2z_5 + \beta)^2}{h_1(\lambda_f - 1)}.$$

Since the map preserves the Hamiltonians the above three expressions are (in that order) also equal to

$$\frac{z_3 + z_5}{\tilde{z}_3 + \tilde{z}_5}, \qquad \frac{z_2 - z_4}{\tilde{z}_2 - \tilde{z}_4}, \qquad \frac{z_1 - z_6}{\tilde{z}_1 - \tilde{z}_6},$$

so that the above equations can be solved linearly in terms of the variables $\tilde{z}_i$. The Poisson matrix of a Poisson structure for this system is given by

$$\begin{pmatrix} 0 & z_6 & -z_5 & 0 & -z_3 & z_2 - 2z_5 \\ -z_6 & 0 & 0 & z_6 - 2z_3 & 0 & -z_1 - z_4 \\ z_5 & 0 & 0 & -z_5 & 0 & 0 \\ 0 & 2z_3 - z_6 & z_5 & 0 & z_3 & -z_2 \\ z_3 & 0 & 0 & -z_3 & 0 & 0 \\ 2z_5 - z_2 & z_1 + z_4 & 0 & z_2 & 0 & 0 \end{pmatrix}.$$

If λ depends on the Casimirs of this Poisson structure only, then the above map is a Poisson map, so it is a Bäcklund transformation; moreover it has the spectrality property. This can be verified directly by computing the brackets $\{\tilde{z}_i, \tilde{z}_j\}$ and verifying the compatibility relations. Alternatively one uses the fact that the map which is induced on the triples of polynomials $(u(x), v(x), w(x))$, as above, is a Bäcklund transformation for an a.c.i. system obtained by removing in the Mumford system the restriction that the polynomial w be monic (the Poisson structures are obtained from (16) by replacing $\sigma \otimes \sigma$ with $\bar{w} - \sigma \otimes \sigma$, where $\bar{w}$ denotes the leading coefficient of $w(x)$). It suffices then to verify that the map which sends $(z_1, \ldots, z_6)$ to $(u(x), v(x), w(x))$ is a Poisson map and has the spectrality property when one takes on the target space the Poisson structure corresponding to $\varphi(x) = x(x-1)$.

3.5. The Hénon–Heiles potential

In this section we show an example how one gets Bäcklund transformations for a.c.i. systems whose generic level set of the integrals is a finite cover of a Jacobian. We do this by lifting the Bäcklund transformation for the underlying family of Jacobians to the cover; since such a lifting is not unique we get, in general, an implicitly defined correspondence, rather than an explicit map.

We treat the case of the Hénon–Heiles system, which is given by the following Hamiltonian on $\mathbb{C}^4$, equipped with the standard symplectic structure,

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + 8q_2^3 + 4q_1^2 q_2.$$

A first integral is given by

$$F = -q_2 p_1^2 + q_1 p_1 p_2 + q_1^2(q_1^2 + 4q_2^2).$$

We use the map defined by

$$u(x) = x^2 - 2q_2 x - q_1^2, \qquad v(x) = \frac{\mathrm{i}}{\sqrt{2}}(p_2 x + q_1 p_1),$$
$$w(x) = x^3 + 2q_2 x^2 + (q_1^2 + 4q_2^2)x - \frac{p_1^2}{2}, \tag{75}$$

which is a morphism to the Mumford system, the latter being equipped with the Poisson structure corresponding to $\varphi(x) = x$. It follows from the results of Section 2 that for any constant $\lambda \in \mathbb{C}$ we get a Bäcklund transformation, given by $\tilde{L} = \mathrm{MLM}^{-1}$, where

$$L(x) = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{2}}(p_2 x + q_1 p_1) & x^3 + 2q_2 x^2 + (q_1^2 + 4q_2^2)x - \frac{p_1^2}{2} \\ x^2 - 2q_2 x - q_1^2 & -\frac{\mathrm{i}}{\sqrt{2}}(p_2 x + q_1 p_1) \end{pmatrix},$$

and

$$M(x) = \begin{pmatrix} \beta & x - \lambda_f + \beta^2 \\ 1 & \beta, \end{pmatrix},$$

where

$$\beta = \frac{\sqrt{2}\mu_f - \mathrm{i}(p_2\lambda_f + q_1 p_1)}{\sqrt{2}(\lambda^2 - 2q_2\lambda - q_1^2)}.$$

Also $\mu_f^2 = f(\lambda_f)$ with

$$f(x) = u(x)w(x) + v^2(x) = x(x^4 - hx - g),$$

where h and g are the values of H and G at the point (q_1, q_2, p_1, p_2). Poissonicity and spectrality are a consequence of the fact that the map $(q_1, q_2, p_1, p_2) \to (u, v, w)$, given by (75) is a Poisson map. One notices that in this case one does not get explicit formulas for $\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2$ but for $\tilde{q}_1^2, \tilde{q}_2, \tilde{q}_1 p_1, \tilde{p}_2$, which stems from the fact that the generic level manifolds of the integrals are 2:1 unramified covers of Jacobians.

4. Concluding remarks

We have constructed Bäcklund transformations for a large class of integrable systems. Basically, we have considered four large families of integrable systems that are of interest in mathematical physics. Indeed, if we choose the following parameterization of the generators (h_j, e_j, f_j) of a direct sum of g or $g+1$ copies of the Lie–Poisson algebra of $\mathfrak{sl}(2)$, in terms of the canonical Darboux variables (coordinates and momenta), (p_j, q_j), $\{p_j, q_k\} = \delta_{jk}$:

$$h_j = \frac{1}{2}\, p_j q_j, \qquad f_j = \frac{1}{2}\, q_j^2, \qquad e_j = -\frac{1}{2} p_j^2 + \frac{2C_j}{q_j^2},$$

then we deal with the following Hamiltonian systems:

(1) In the case of the Mumford system the Hamiltonian is of the form

$$H = \frac{1}{2}\sum_{i=1}^{g} p_i^2 - \sum_{i=1}^{g} \frac{2C_i}{q_i^2} - \frac{1}{2}\sum_{i=1}^{g} q_i^2(a_i + C_0) + \frac{1}{4}\left(\sum_{k=1}^{g} q_k^2\right)^2,$$

so this case is a generalization of the g-dimensional Garnier system.

(2) For the even Mumford system the Hamiltonian function describes the motion of a particle in a potential of order 6:

$$H = \frac{1}{2}\sum_{i=1}^{g} p_i^2 - \sum_{i=1}^{g} \frac{2C_i}{q_i^2} - \frac{1}{2}\sum_{i=1}^{g}(a_i^2 + a_i C_{-1} + C_0) q_i^2$$
$$+ \frac{1}{4}\left(\sum_{k=1}^{g} q_k^2\right)\sum_{i=1}^{g}(C_{-1} + 2a_i) q_i^2 - \frac{1}{8}\left(\sum_{k=1}^{g} q_k^2\right)^3.$$

(3) In the odd generalized case we have an integrable system with linear potential

$$H = \frac{1}{2}\sum_{i=0}^{g} p_i^2 - \sum_{i=1}^{g} \frac{2C_i}{q_i^2} + \frac{1}{2} q_0.$$

(4) In the even generalized case we have a g-dimensional harmonic oscillator

$$H = \frac{1}{2}\sum_{i=0}^{g} p_i^2 - \sum_{i=0}^{g} \frac{2C_i}{q_i^2} - \frac{1}{2}\sum_{i=0}^{g} q_i^2.$$

In other words we have showed how to construct in a systematic way Bäcklund transformations for integrable systems linearizable on hyperelliptic Jacobians or generalized hyperelliptic Jacobians. Since for many classical integrable systems it is known how to embed them into Mumford systems [21], our construction produces many new integrable discretizations of Liouville integrable systems, such as the Kowalevski, Goryachev–Chaplygin and Euler tops, Toda lattices and the Gaudin magnet.

For further reading see [6,13].

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